Hilbert, Gödel, and Metamathematics Today

Jeremy Avigad

Department of Philosophy and Department of Mathematical Sciences
Carnegie Mellon University

April, 2011

Overview

An outline of this talk:

- Hilbert, Gödel, and the metamathematical tradition
  - Hilbert's program and metamathematics
  - Gödel and the metamathematical tradition
  - Gödel's remarks on finitism and syntax
  - Interpreting Gödel's ambivalence
- Metamathematics today
  - Proof mining
  - Automated reasoning and formal verification
  - Combinatorial independences
  - History and philosophy of mathematics

Hilbert's program

Foundational works leading up to the program:

- *Grundlagen der Geometrie*, 1899
- *Über den Zahlbegriff*, 1900
- *Mathematische Probleme*, 1900
- *Über die Grundlagen der Logik und der Arithmetik*, 1904
- *Axiomatisches Denken*, 1918
- *Neubegründung der Mathematik*, 1922
- *Die logischen Grundlagen der Mathematik*, 1923

Hilbert's program

Core methodological presuppositions:

- Formal axiomatic systems provide faithful representations of mathematical argumentation.

- With these representations, at least *some* foundational and epistemological questions can be formulated in mathematical terms.

- A finitary, syntactic perspective makes it possible to address such questions without presupposing substantial portions of the body of mathematics under investigation.

An overview of Gödel's work

- Early metamathematical work
  - Completeness and compactness (1929)
  - The incompleteness theorems (1931)
  - Decidability and undecidability for fragments of FOL (1932, 1933)
  - Intuitionistic logic and the double-negation translation (1932, 1933)
  - The provability interpretation of intuitionistic logic (1933)
  - The Dialectica interpretation (1941/1958)
- Set theory
  - The consistency of the axiom of choice and the continuum hypothesis (1938)
- Foundations and philosophy of physics
  - Rotating models of the field equations (1949)
- The philosophy of logic and mathematics (ongoing)

Gödel on his methods

The proof of the above theorems is constructive in the sense that, if a contradiction were obtained in the enlarged system, a contradiction in T could actually be exhibited. (abstract, 1938)

In conclusion, let me make the remark about the *means of proof* used in what follows. Concerning them, no restriction whatsoever has been made. In particular, essential use is made of the principle of the excluded middle for infinite collections (the nondenumerable infinite, however, is not used in the main proof). (dissertation, 1929)

Weyl's obituary for Hilbert (1944)

When one inquires into the dominant influences acting upon Hilbert in his formative years one is puzzled by the peculiarly ambivalent character of his relationship to Kronecker: dependent on him, he rebels against him. Kronecker's work is undoubtedly of paramount importance for Hilbert in his algebraic period. But the old gentleman in Berlin, so it seemed to Hilbert, used his power and authority to stretch mathematics upon the Procrustean bed of arbitrary philosophical principles and to suppress such developments as did not conform.. . . A late echo of this old feud is the polemic against Brouwer's intuitionism with which the sexagenarian Hilbert opens his first article on "Neubegründung der Mathematik" (1922): Hilbert's slashing blows are aimed at Kronecker's ghost whom he sees rising from the grave. But inescapable ambivalence even here — while he fights him, he follows him: reasoning along strictly intuitionistic lines is found necessary by him to safeguard non-intuitionistic mathematics.

Gödel on Hilbert's program

I wish to note expressly that [the second incompleteness theorem for the formal systems under consideration] do not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used, and it is conceivable that there exist finitary proofs that *cannot* be expressed [in the relevant formalisms]. (1930)

I would like to remark by the way that Gentzen sought to give *a "proof"* of this rule of inference and even said that this was the essential part of his consistency proof. In reality, it's not a matter of proof at all, but of an appeal to evidence...I think it makes more sense to formulate an axiom precisely and to say that it is just not further reducible. But here again the drive of Hilbert's pupils to derive something from nothing stands out. (1938)

Gödel on mathemathematics and finitary reasoning

This blindness (or prejudice, or whatever you may call it) of logicians is indeed surprising. But I think the explanation is not hard to find. It lies in a widespread lack, at that time, of the required epistemological attitude towards metamathematics and toward non-finitary reasoning.

Non-finitary reasoning in mathematics was widely considered to be meaningful only to the extent to which it can be "interpreted" or "justified" in terms of a finitary metamathematics. (Note that this, for the most part, has turned out to be impossible in consequence of my results and subsequent work.) (letter to Hao Wang in 1967)

Gödel on mathemathematics and finitary reasoning

I would like to add that there was another reason which hampered logicians in the application to metamathematics, not only of transfinite reasoning, but of mathematical reasoning in general. It consists in the fact that, largely, metamathematics was not considered as a science describing objective mathematical states of affairs, but rather as a theory of the human activity of handling symbols. (letter to Wang, 1968)

Those who, like Carnap, misuse symbolic language want to discredit mathematical logic; they want to prevent the appearance of philosophy. The whole movement of the positivists want to destroy philosophy; for this purpose, they need to destroy mathematical logic as a tool. (to Wang, 1972)

Gödel's assessment of Hilbert's program (1961)

skepticism   spiritualism
materialism   idealism
positivism   theology

Gödel's assessment of Hilbert's program (1961)

Although the nihilistic consequences are very well in accord with the spirit of the time, here a reaction set in—obviously not on the part of philosophy, but rather on that of mathematics, which, by its nature, as I have already said, is very recalcitrant in the face of the *Zeitgeist*. And thus came into being that curious hybrid *[merkwürdige Zwitterding]* that Hilbert's formalism represents, which sought to do justice both to the spirit of the time and the nature of mathematics. It consists in the following: on the one hand, in conformity with the ideas prevailing in today's philosophy, it is acknowledged that the truth of the axioms from which mathematics starts out cannot be justified or recognized in any way, and therefore the drawing of consequences from them has meaning only in a hypothetical sense, whereby this drawing of consequences itself (in order to satisfy even further the spirit of the time) is construed as a mere game with symbols according to certain rules, likewise not [supported by] insight. (Undelivered lecture for the APS, around 1961)

Gödel's assessment of Hilbert's program (1961)

The correct attitude appears to me to be that the truth lies in the middle, or consists of a combination of the two conceptions.

Now, in the case of mathematics, Hilbert has of course attempted such a combination, but one obviously too primitive and tending too strongly in one direction.

Hilbert and Gödel: a comparison

What they had in common: faith in the ability rational inquiry to address fundamental questions.

Hilbert: ultimate faith in the ability of mathematics to solve all problems; skeptical of philosophy; aimed to replace philosophical questions with properly mathematical ones.

Gödel: sensitive to the limitations of formal methods; looked to philosophy to fill the gap.

"The analysis of concepts is central to philosophy. Science only combines concepts and does not analyze concepts. It contributes to the analysis of concepts by being stimulating for real analysis. . . Analysis is to arrive at what thinking is based on: the inborn intuitions."

Contemporary lessons

The metamathematical approach, involving a syntactic, formal modeling of mathematical methods, provides a powerful means of understanding mathematics.

At the same time, the mathematical modeling should be complemented by a properly philosophical reflection on the subject matter.

Mathematical logic is often most compelling and satisfying when these two strands come together, that is, when mathematical results contribute to a deeper philosophical understanding.

Overview

An outline of this talk:

- Hilbert, Gödel, and the metamathematical tradition
  - Hilbert's program and metamathematics
  - Gödel and the metamathematical tradition
  - Gödel's remarks on finitism and syntax
  - Interpreting Gödel's ambivalence
- Metamathematics today
  - Proof mining
  - Automated reasoning and formal verification
  - Combinatorial independences
  - History and philosophy of mathematics

Proof mining

For most of its history, mathematics has been explicitly computational.

Developments in the nineteenth century inaugerated methods that are "infinitary," "abstract," "conceptual," "nonconstructive," "set-theoretic," "impredicative," and so on. These tend to suppress (or eliminate completely) quantitative and computational information.

Hilbert's program, while outwardly focused on questions as to the consistency of the new methods, can be more broadly viewed as trying to recapture explicit and computational meaning.

Metamathematical research in the twentieth century provides various means of "foundational reduction."

Proof mining

Proof mining today aims to put these insights to good use, for example, by:

1. Extracting explicit quantitative information from abstract / infinitary / nonconstructive mathematical arguments (such as numerical bounds, rates of convergence).

2. Extracting mathematically useful *uniformities* in analysis.

3. Developing infinitary methods better suited to finite combinatorial problems.

4. Interpreting nonconstructive developments (for example, in commutative algebra) in algorithmic, computational terms.

Proof mining

The big questions:

- What is gained by the use of infinitary methods in mathematics?
- What is lost?
- To what extent can we maximize the gains while minimizing the losses?

Formal verification and automated reasoning

"Formal verification" is a branch of computer science that uses formal methods to verify correctness:

- of hardware and software design (relative to specifications)
- of mathematical proof

Interactive proof assistants now help construct formal axiomatic proofs:

- Keeping track of libraries of definitions and previous proofs.
- Parsing mathematical language and filling in implicit information.
- Filling in reasoning steps automatically.
- Performing detailed calculations (and verifying them).

All of these require a better understanding of mathematical language and method.

Formal verification and automated reasoning

Some mathematical theorems that have been verified to date:

- Gödel's first incompleteness theorem (Shankar 1986; O'Connor, 2004)

- The prime number theorem (Avigad, 2004; Harrison, 2008)

- The four-color theorem (Gonthier, 2004)

- The Jordan curve theorem (Hales, 2005)

- Dirichlet's theorem on primes in an arithmetic progression (Harrison, 2009)

Some ambitious projects underway:

- Verification of the Feit-Thompson theorem (Gonthier)

- Verification of a proof of the Kepler conjecture (Hales)

- "Univalent" foundations for algebraic topology (Voevodsky)

Formal verification and automated reasoning

The big questions:

- How does mathematical language work?

- How does mathematical inference work?

- How do we represent / store / access / use mathematical knowledge?

- How do the methods of mathematics make it possible for us to understand complex mathematical arguments?

Combinatorial independences

For a number of years, Harvey Friedman has sought finitary combinatorial statements that seem to be true (for example, they are provable from strong axioms regarding the infinite), but are not provable without strong axioms.

On the other hand, a good deal of research (for example, in Reverse Mathematics) shows that "ordinary" mathematical methods do not require such strong assumptions.

The big questions:

- What role does the infinite play in ordinary mathematical arguments?
- What bearing do infinitary assumptions have on our understanding of the finite?

History and philosophy of mathematics

Traditionally, the philosophy of mathematics has focused on issues of justification and correctness.

But in ordinary mathematical usage, one finds a much richer array of value judgments: questions can be natural, theorems can be striking, concepts can be fruitful, theories can be profound, some proofs explain better than others.

These raise questions as to the nature of mathematical *understanding*.

Studying the history of mathematics yields important insights.

Philosophical analysis should inform and be informed by a formal understanding.

Conclusions

We have entered the "golden age of metamathematics."

- After a century of lively development, basic concepts have settled and methods have matured.
- We are making progress on fundamentally important questions.