Towards a Modal Proof Theory of Topological Dynamics

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Dynamic Topological Logic

Dynamic topological systems: X a topological space, $f : X \to X$ continuous.

Language (L):

$$p \qquad \neg\varphi \qquad \varphi \wedge \psi \qquad \Box\varphi \qquad (f)\varphi \qquad [f]\varphi$$

Semantics: $[\![\cdot]\!] : \mathsf{L} \to \mathcal{P}(X)$

- $[\![\Box\varphi]\!] = [\![\varphi]\!]^\circ$ (interior)
- $[\![(f)\varphi]\!] = f^{-1}\,[\![\varphi]\!]$ (next)
- $[\![[f]\varphi]\!] = \bigcap_{n<\omega} f^{-n}\,[\![\varphi]\!]$ (henceforth)

Origins

- The topological interpretation of S4 was already known by McKinsey and Tarski in the 1940's
- Artemov, Davoren, Nerode add dynamics and study the logic with $\Box$ and (f) in the 1990's.
- Kremer and Mints add the 'henceforth' operator in the 2000's.

The logic of Minimal Systems

Minimal systems: Defined as dynamic topological systems $\langle X, f\rangle$ which contain no proper, topologically closed, f-invariant subsystems.

Equivalent condition: The orbit of every point is dense.

This is expressible in $\mathsf{L}_\forall$ by

$$\exists\Box p \to \forall\langle f\rangle p.$$

$\mathcal{DTL}$(Min) is:

- Equal to $\mathcal{DTL} + \exists\Box p \to \forall\langle f\rangle p$.
- Decidable, but not in primitive recursive time[1].
- No finite or even locally finite model property.
- Finite quasimodel property.

[1] Bound adapted from Konev, Kontchakov, Wolter and Zakharyaschev.

Recursive enumerability

Theorem (DFD)

The set of valid formulas of $\mathcal{DTL}$ is recursively enumerable.

Proof idea.

Three basic steps to verify that φ is valid:

1. Simulate dynamic topological models by finite Kripke models.
2. Use these to build an increasing sequence of partial quasimodels converging to a limit model where φ is false.
3. Otherwise, use Kruskal's theorem to give an effective termination criterion[2]. □

[2] Also adapted from Konev, Kontchakov, Wolter and Zakharyaschev.

Kremer-Mints axioms

Taut		All propositional tautologies.
Axioms for $\Box$:		
	K	$\Box(p \to q) \to (\Box p \to \Box q)$
	T	$\Box p \to p$
	4	$\Box p \to \Box\Box p$
Temporal axioms:		
	$\mathrm{Neg}_{(f)}$	$\neg(f)p \leftrightarrow (f)\neg p$
	$\mathrm{And}_{(f)}$	$(f)(p \wedge q) \leftrightarrow (f)p \wedge (f)q$
	$\mathrm{Fix}_{[f]}$	$[f]p \to p \wedge (f)[f]p$
	$\mathrm{Ind}_{[f]}$	$[f](p \to (f)p) \to (p \to [f]p)$
Cont		$(f)\Box p \to \Box(f)p$.
Rules:		
	MP	Modus ponens
	Subs	Substitution
	$\mathrm{N}_{\Box}$	Necessitation for $\Box$
	$\mathrm{N}_{(f)}$	Necessitation for (f)
	$\mathrm{N}_{[f]}$	Necessitation for $[f]$

Simulations

Definition

A simulation is a continuous binary relation χ between dynamic topological models $\mathfrak{X}, \mathfrak{Y}$ such that, if $x \; \chi \; y$ and p is a propositional variable, then $x \in [\![p]\!]_{\mathfrak{X}}$ if and only if $y \in [\![p]\!]_{\mathfrak{Y}}$.

We write $\langle \mathfrak{X}, x \rangle \trianglelefteq \langle \mathfrak{Y}, y \rangle$ if and only if there exists a simulation χ with $x \; \chi \; y$.

A formula φ defines being simulated by $\langle \mathfrak{X}, x \rangle$ if, for every dtm $\mathfrak{Y}$ and every $y \in Y$,

$$\langle \mathfrak{Y}, y \rangle \models \varphi \Leftrightarrow \langle \mathfrak{X}, x \rangle \trianglelefteq \langle \mathfrak{Y}, y \rangle \, .$$

Theorem (DFD)

There exists a finite topological model $\mathfrak{W}$ and $w \in W$ such that the property of being simulated by $\langle \mathfrak{W}, w \rangle$ is not definable in L.

The tangled closure

Definition

If $\mathcal{S}$ is a collection of subsets of a topological space X, we defne the tangled closure of $\mathcal{S}$, denoted $\mathcal{S}^*$, as the greatest subset of X such that every $A \in \mathcal{S}$ is dense in $\mathcal{S}^*$.

L*: We consider an extension of L where $\Diamond$ is allowed to act on *sets* of formulas, and define

$$[\![\Diamond(\gamma_0, ..., \gamma_n)]\!] = \{[\![\gamma_0]\!], ..., [\![\gamma_n]\!]\}^* .$$

A similar extension of the modal language was first considered by Dawar and Otto.

Theorem (DFD)

Given a finite topological model $\mathfrak{W}$ and $w \in W$, there exists a formula $\mathrm{Sim}(w) \in \mathsf{L}^*_\Diamond$ such that

$$\langle \mathfrak{X}, x \rangle \models \mathrm{Sim}(w) \Leftrightarrow \langle \mathfrak{W}, w \rangle \trianglelefteq \langle \mathfrak{X}, x \rangle .$$

A complete axiomatization for $\mathcal{DTL}^*$

Taut		All propositional tautologies.
Axioms for $\Box$:		
	K	$\Box(p \to q) \to (\Box p \to \Box q)$
	T	$\Box p \to p$
	4	$\Box p \to \Box\Box p$
	$\mathrm{Fix}_{\Diamond}$	$\Diamond\Gamma \to \bigwedge_{\gamma\in\Gamma} \Diamond(\gamma \wedge \Diamond\Gamma)$
	$\mathrm{Ind}_{\Diamond}$	$p \wedge \Box\Big(p \to \bigwedge_{\gamma\in\Gamma} \Diamond(p \wedge \gamma)\Big) \to \Diamond\Gamma$
Temporal axioms:		
	$\mathrm{Neg}_{(f)}$	$\neg(f)p \leftrightarrow (f)\neg p$
	$\mathrm{And}_{(f)}$	$(f)(p \wedge q) \leftrightarrow (f)p \wedge (f)q$
	$\mathrm{Fix}_{[f]}$	$[f]p \to p \wedge (f)[f]p$
	$\mathrm{Ind}_{[f]}$	$[f](p \to (f)p) \to (p \to [f]p)$
TCont		$(f)\Box\Gamma \to \Box(f)\Gamma$
Rules:		MP, Subs, $\mathrm{N}_{\Box}$, $\mathrm{N}_{[f]}$, $\mathrm{N}_{(f)}$

Applications

Conjecture

$\mathcal{DTL}^*_\forall(\mathrm{Min}) = \mathcal{DTL}^*_\forall + \exists\Box p \to \forall\langle f \rangle p$

It may now be possible to axiomatize

- $\mathcal{DTL}^*$ of complete metric spaces
- $\mathcal{DTL}^*$ of Aleksandroff spaces (essentially Kripke models)
- $\mathcal{DTL}^*$ of the plane

$\mathcal{DTL}+$ probability?

Definition

A *probability-preserving system* is a probability space $\langle X, \mu \rangle$ equipped with a function f such that $\mu = \mu f^{-1}$.

Theorem (Poincaré)

On a probability-preserving system, if $\mu(A) > 0$, then A contains a recurrent point x; that is, such that, for some $n > 0$, $f^n(x) \in A$.

A topological measure space in addition has a topology such that all non-empty opens have positive measure.

Here we can express Poincaré Recurrence by

$$\Box p \to \Diamond(f)\langle f \rangle p.$$

Measure algebras

For probability-preserving systems, it is natural to disregard sets of probability zero.

Scott: S4 can be interpreted over measure algebras, the quotient $\mathcal{P}(X)/\ker(\mu)$.

Theorem (DFD, Lando)

S4 *is complete for interpretations on the measure algebra of the unit interval.*

Question: *What is the* $\mathcal{DTL}$ *of the measure algebra of probability-preserving systems?*

Thank You!