Towards a New Computational Interpretation of Sub-classical Principles

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Delimited Control Operators

- Coming from Theoretical Computer Science
- Very abstract way to capture *computational effects* from the theory of programming languages
- Starting interest: essential ingredient in a likely constructive proof of completeness of intuitionistic logic w.r.t. Kripke models, due to Olivier Danvy
- But, no simple/logical explanation
- Topic of PhD thesis supervised by Hugo Herbelin

- Type-and-effect systems (Danvy-Filinski 1991)
  - hard to relate to traditional logic – implication a quaternary connective
- Classical logic
  - can be considered as computational contents of classical logic
  - but, classical logic does not have the Disjunction and Existence Properties – not nice for a programming language
  - also, computationally more powerful than older (non-delimited) control operators that are behind some computational interpretations of classical logic
- Intuitionistic logic (Herbelin 2010)
  - one can use them to derive Markov's Principle
  - but, one does not lose Disjunction and Existence Properties

Delimited Control Operators

Example 1 – use as “exceptions”

$$1 + \# (2 + \mathcal{S}\, k.4)$$

Delimited Control Operators

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\begin{aligned}
& 1 + \#(2 + \mathcal{S}\,k.4) \\
\rightarrow\ & 1 + \#4
\end{aligned}
$$

Delimited Control Operators

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$$
\begin{aligned}
& 1 + \#\,(2 + \mathscr{S}\, k.4) \\
\rightarrow & 1 + \#4 \\
\rightarrow & 1 + 4 \\
\rightarrow & 5
\end{aligned}
$$

Delimited Control Operators

Example 2 – full use

$$1 + \#\,(2 + \mathscr{S}\,k.k(k4))$$

Delimited Control Operators

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& 1 + \#\,(2 + \mathscr{S}\,k.k(k4)) \\
\rightarrow\ & 1 + \#\,((\lambda a.\#(2+a))\,((\lambda a.\#(2+a))4))
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Delimited Control Operators

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Delimited Control Operators

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\rightarrow^{+} & 9
\end{aligned}
$$

MQC$^+$– Intuitionistic Extension of Intuitionistic Logic

To the natural deduction system of minimal intuitionistic logic, add:

1. A rule marking that a $\{\Rightarrow, \forall\}$-free formula T is being proved:

$$\frac{\Gamma \vdash_T \quad T}{\Gamma \vdash \quad T}$$

2. A rule for using classical logic in the delimited derivation sub-tree:

$$\frac{\Gamma, \quad A \Rightarrow T \vdash_T \quad T}{\Gamma \vdash_T \quad A}$$

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$$\frac{\Gamma, k : A \Rightarrow T \vdash_T p : T}{\Gamma \vdash_T \mathcal{S} k.p : A}$$

Proof terms:

$$p, q, r ::= a \mid \iota_1 p \mid \iota_2 p \mid \mathsf{case}\ p\ \mathsf{of}\ \left(a.q \| b.r\right) \mid (p, q) \mid \pi_1 p \mid \pi_2 p \mid \lambda a.p \mid$$
$$\mid pq \mid \lambda x.p \mid pt \mid (t, p) \mid \mathsf{dest}\ p\ \mathsf{as}\ (x.a)\ \mathsf{in}\ q \mid \#p \mid \mathscr{S} k.p$$

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Pure evaluation contexts:

$$P ::= [\,] \mid \mathsf{case}\ P\ \mathsf{of}\ \left(a_1.p_1 \| a_2.p_2\right) \mid \pi_1 P \mid \pi_2 P \mid \mathsf{dest}\ P\ \mathsf{as}\ (x.a)\ \mathsf{in}\ p \mid$$
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MQC$^+$– Intuitionistic Extension of Intuitionistic Logic

Extending the λ-Calculus of Proof Terms

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$$Pq \mid (\lambda a.q)P \mid Pt \mid \iota_1 P \mid \iota_2 P \mid (P, p) \mid (V, P) \mid (t, P)$$

Reduction: (Call-by-value strategy)

$$(\lambda a.p)V \to p\{V/a\} \qquad \mathsf{case}\ \iota_i V\ \mathsf{of}\ \left(a_1.p_1 \| a_2.p_2\right) \to p_i\{V/a_i\}$$
$$(\lambda x.p)t \to p\{t/x\} \qquad \mathsf{dest}\ (t, V)\ \mathsf{as}\ (x.a)\ \mathsf{in}\ p \to p\{t/x\}\{V/a\}$$
$$\pi_i(V_1, V_2) \to V_i \qquad \# P[\mathcal{S}\, k.p] \to \# p\{(\lambda a.\# P[a]) / k\}$$
$$\# V \to V \qquad E[p] \to E[p'] \text{ when } p \to p'$$

Markov's Principle (predicate logic version):

$$\neg\neg S \Rightarrow S, \qquad \text{for } S \text{ a } \{\Rightarrow, \forall\}\text{-free-formula}$$

$$\lambda a.\#\bot_E(a(\lambda b.\, \mathscr{S} k.b))$$

MQC$^+$– Examples of Derivations

Deriving the predicate logic version of Double-negation Shift

Double Negation Shift (predicate logic version):

$$\forall x(\neg\neg A(x)) \Rightarrow \neg\neg(\forall x A(x))$$

$$\lambda a.\lambda b.\#b(\lambda x.\,\mathcal{S}k.axk)$$

A structure $(K, \leq, D, \Vdash^s_{,} \perp\!\!\!\perp)$, where:

- $(K, \leq)$ a preorder of *worlds*;
- $w \perp\!\!\!\perp^a C$ a relation labelling a world w as *exploding* at formula C and annotation a;
- $w \Vdash^s_a A_0$ a relation of *strong forcing* of atomic formulae A_0, such that
$$\text{for all } w' \geq w, w \Vdash^s_a A_0 \to w' \Vdash^s_a A_0,$$
- $D(w)$, a *domain of quantification* for each world w, such that
$$\text{for all } w' \geq w, D(w) \subseteq D(w').$$
- and, ...

Kripke-CPS Models

The Semantic Side of MQC^+

- and, strong forcing $\Vdash^s$ is *extended from atomic to composite formulae* inductively and by simultaneously defining a new relation, *(non-strong) forcing* $\Vdash$:

$$w \Vdash_a A = \begin{cases} \forall w_1 \geq w.\ \left(\forall w_2 \geq w_1.\ w_2 \Vdash^s_T A \to w_2 \perp\!\!\!\perp^T T\right) \to w_1 \perp\!\!\!\perp^T T & , a = T \\ \forall C.\ \forall w_1 \geq w.\ \left(\forall w_2 \geq w_1.\ w_2 \Vdash^s A \to w_2 \perp\!\!\!\perp C\right) \to w_1 \perp\!\!\!\perp C & , \text{no } a \end{cases}$$

$$w \Vdash^s_a A \wedge B \text{ if } w \Vdash_a A \text{ and } w \Vdash_a B$$

$$w \Vdash^s_a A \vee B \text{ if } w \Vdash_a A \text{ or } w \Vdash_a B$$

$$w \Vdash^s_a A \Rightarrow B \text{ if for all } w' \geq w, w \Vdash_a A \text{ implies } w \Vdash_a B$$

$$w \Vdash^s_a \forall x.A(x) \text{ if for all } w' \geq w \text{ and all } t \in D(w'), w' \Vdash_a A(t)$$

$$w \Vdash^s_a \exists x.A(x) \text{ if } w \Vdash_a A(t) \text{ for some } t \in D(w)$$

Theorem (Soundness)

If $\Gamma \vdash_a A$*, then for all* $w, a' \geq a$, $w \Vdash_a \Gamma$ *implies* $w \Vdash_a A$.

Theorem (Completeness for $\mathscr{U}$)

There is a "universal" model $\mathscr{U}$ *s.t. for any world* $\Gamma \in \mathscr{U}$*, if* $\Gamma \Vdash_a A$*, then* $\Gamma \vdash_a^{nf} A$.

Corollary (Normalisation of MQC$^+$)

$\vdash A \mapsto \ \vdash^{nf} A$

Corollary (DP and EP for MQC$^+$)

If $\vdash A \vee B$*, then either* $\vdash A$ *or* $\vdash B$.

If $\vdash \exists x A(x)$*, then there exists* t *such that* $\vdash A(t)$.

Theorem (Glivenko 1929)

For propositional logic, $\vdash^i \neg\neg A \longleftrightarrow \vdash^c A$

Theorem (Gödel 1933)

For predicate logic, $\vdash^i A^\perp \longleftrightarrow \vdash^c A$

Theorem (Glivenko's Theorem for Predicate Logic)

$\vdash^+ \neg\neg A \longleftrightarrow DNS \vdash^i A^\perp \longleftrightarrow \vdash^c A$

Work directions:

1. A system unifying current variants
2. Constructive Reverse Mathematics
3. Constructive proofs of completeness
4. Revisiting *Dialectica* and Bar Recursion
5. Computational Effects

1. A System Unifying Current Approaches

- The shown system can prove DNS (arbitrarily many instances)
- But, only one instance of MP
- And no instances of MP if DNS is also proved
- Herbelin's system can prove arbitrarily many instances of MP
- But, no instance of DNS

2. Constructive Reverse Mathematics

- Investigating already existing connections in Intuitionistic Reverse Mathematics
- Solovay-Moschovakis: $\mathrm{BI} + \mathrm{MP} \vdash \mathrm{DNS}$
- Veldman: Completeness for Kripke models using FAN
- Veldman: $\mathrm{MP} \vdash \mathrm{OIP} \Leftrightarrow \mathrm{DNS}$
- Ishihara: variants of MP

3. Constructive proofs of completeness

- Gödel's proof of Completeness is essentially constructive
- Proved by Krivine using a double-negation translation
- Should be possible to prove it directly: delimited control operators are precisely a way to unwind double-negation proofs

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- No a priori constructive proof of Completeness for intuitionistic logic w.r.t. Kripke semantics, when one considers all logical connectives
- Proved by Wim Veldman using the Fan theorem
- Should be possible to prove it directly: such an algorithm was already given by Olivier Danvy

4. Revisiting *Dialectica* and Bar Recursion

- Gödel's *Dialectica* interpretation validates MP
- Spector's extension validates DNS and thus the double-negation translation of Countable Choice
- More recent version of bar recursion by Coquand, Berger, Kohlenbach
- Can delimited control operators be used as (less complex) alternative?

5. Computational Effects

- Delimited control operators were invented in Semantics of programming languages
- They can express any monadic computational effect (Filinski 1994)
- Logical explanation of computational effects an open problem
- Also, reduction for delimited control operators so far specified as weak-head subset
- Normalisation via Kripke-CPS models gives a way to find out the full reduction relation