Proof Interpretations and Their Application to Current Mathematics

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Our approach is based on novel forms and extensions of:

K. Gödel's functional ('Dialectica') interpretation!

Historically proof interpretations $\mathcal{I}$ were used for **consistency proofs**: usually $T^{\mathcal{I}}$ can be proved an a more elementary quantifier-free system $\mathcal{T}_{qf}$, than the system $\mathcal{T}$ used to prove T.

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this reduces the consistency problem of $\mathcal{T}$ to that of $\mathcal{T}_{qf}$.

G. Kreisel (1951,...): use $\mathcal{I}$ to **extract new information** from interesting proofs of existential statements.

Proof interpretations as tool for generalizing proofs

$$\begin{array}{ccccc} P & \xrightarrow{\mathcal{I}} & P^{\mathcal{I}} & & \\ G \downarrow & & \downarrow \mathcal{I}^G & & \\ P^G & \xrightarrow{G^{\mathcal{I}}} & (P^{\mathcal{I}})^G & = & (P^G)^{\mathcal{I}} \end{array}$$

- Generalization $(P^{\mathcal{I}})^G$ of $P^{\mathcal{I}}$: **easy** since $(P^{\mathcal{I}})^G$ **is finitary!**

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T. Tao: $P =$ **'soft analysis'**, $P^{\mathcal{I}} =$ **'hard or finitary analysis'**.

Example: the monotone convergence principle

Let (a_n) be a nonincreasing sequence in $[0,1]$. Then, clearly, (a_n) is convergent and so a Cauchy sequence which we write as:

$$\mathbf{(1)}\ \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \in \mathbb{N} \forall i,j \in [n; n+m]\ (|a_i - a_j| \leq 2^{-k}),$$

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Consider the (partial) Herbrand normal form of this statement is

$$\mathbf{(2)}\ \forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i,j \in [n; n+g(n)]\ (|a_i - a_j| \leq 2^{-k}).$$

In contrast to (1), there is a **simple (primitive recursive) bound** $\Phi^*(g, k)$ on (2) (Kreisel's **no-counterexample interpretation** also referred to as **'metastability'** by T.Tao):

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Proposition (G. Kreisel 1951)

Let (a_n) be any nonincreasing sequence in $[0,1]$ then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \leq \Phi^*(g,k) \forall i,j \in [n; n+g(n)] \, (|a_i - a_j| \leq 2^{-k}),$$

where

$$\Phi^*(g,k) := \tilde{g}^{(2^k)}(0) \text{ with } \tilde{g}(n) := n + g(n).$$

Moreover, there exists an $i < 2^k$ such that n can be taken as $\tilde{g}^{(i)}(0)$.

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- For more complicated formulas the latter is **much more involved** (already for the infinitary pigeonhole principle; compare again Tao).
- Proper understanding of functional interpretation requires treatment of systems based on **intuitionistic logic** (Brouwer).

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$.

$$\mathbf{A_n(x)} := \frac{\mathbf{1}}{\mathbf{n+1}}\mathbf{S_n(x)}, \text{ where } \mathbf{S_n(x)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^i(x)} \quad (\mathbf{n} \geq \mathbf{0}).$$

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Based on logical metatheorem to be discussed below:

Theorem (Leuştean/K., Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists \mathbf{n} \leq \mathbf{\Phi}(\varepsilon, \mathbf{g}, \mathbf{b}, \eta)\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n} + \mathbf{g}(\mathbf{n})] \left(\|\mathbf{A_i}(\mathbf{x}) - \mathbf{A_j}(\mathbf{x})\| < \varepsilon\right),$$

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where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^K(0), \text{ with}$$
$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$
$$h, \tilde{h} : \mathbb{N} \to \mathbb{N},\ h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \le n} h(i).$$

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Special Hilbert case: treated prior by Avigad/Gerhardy/Towsner (again based on logical metatheorem).

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Answers: Crucial: **no separability assumption** on X was used. Yes, **there are suitable logical metatheorems**.

Many abstract types of metric structures can be added as atoms: metric, hyperbolic, CAT(0), δ-hyperbolic, normed, uniformly convex, Hilbert, ... spaces or $\mathbb{R}$-trees X : add **new base type** X, all **finite types over** $\mathbb{N}, X$ and a new **constant** d_X representing d etc.

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Counterexamples (to extractability of uniform bounds): for the classes of strictly convex ($\rightarrow$ uniformly convex) or separable ($\rightarrow$ totally bounded) spaces!

A formal system for analysis

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map type-ρ objects to type-τ objects.

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$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$+**DC**, where

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$\mathcal{A}^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic ... space.

A novel form of majorization

y, x functionals of types ρ and $\widehat{\rho} := \rho[\mathbb{N}/X]$:

$$\mathbf{x}^{\mathbb{N}} \gtrsim_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
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Example:

$$\mathbf{f}^* \gtrsim_{\mathbf{X} \to \mathbf{X}} \mathbf{f} \equiv \forall \mathbf{n} \in \mathbb{N}, \mathbf{x} \in \mathbf{X}[\mathbf{n} \geq \|\mathbf{x}\| \to \mathbf{f}^*(\mathbf{n}) \geq \|\mathbf{f}(\mathbf{x})\|].$$

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WARNING: Already for $f : X \to X$ only weak rule of extensionality!

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then **monotone functional interpretation** extract a **computable functional** $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ s.t. for all α, b

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$$\big(\mathbf{f} \text{ n.e.} \wedge \|\mathbf{x}\|, \|\mathbf{f}(0)\| \le \mathbf{b} \to \exists \mathbf{v} \le \Phi(\alpha, \mathbf{b}) \mathbf{A}_\exists\big)$$

holds in **all nonempty (real) Hilbert space X.**

As special case of **general logical metatheorems** due to Gerhardy/K. (Trans. Amer. Math. Soc. 2008) one has (K a compact metric space, $A_\exists$ an existential property):

Theorem (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$ proves

$$\forall \alpha \in \mathbb{N}^{\mathbb{N}}\, \forall y \in K\, \forall x \in X\, \forall f : X \to X\, \big(f \text{ n.e.} \to \exists v \in \mathbb{N}\, A_\exists\big),$$

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Uniformly convex case: bound Φ depends on modulus of convexity η.

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'We shall establish Theorem 1.6 by "finitary ergodic theory" techniques, reminiscent of those used in [Green-Tao]...' 'The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit'...'In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation'
(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)

- Since 2000 more than 40 papers with applications of proof theory in nonlinear analysis (Avigad, Briseid, Gerhardy, K., Kreuzer, Lambov, Leustean, Oliva, Safarik, Towsner) published in journals such as: Nonlinear Analysis, J. Math. Anal. Appl., J. of Nonlinear and Convex Analysis, Fixed Point Theory, Numer. Funct. Anal. Optimiz. and general math journals such as Advances in Mathematics, Fundamenta Mathematicae, J. European. Math. Soc., Trans. Amer. Math. Soc.

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- Many new results on the proof theoretic side (Ann. Pure Appl. Logic, Notre Dame J. Logic, Math. Log. Quart., J. Symb. Logic).

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  General metatheorem for this?

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  Find new classes where this is possible!
- Finding of **new areas for proof mining**: e.g. geometric group theory, PDE's, C^*-algebras.

Is there any mathematical principle other than extensionality that puts a limitation to the proof mining program, i.e. that does not preserve any finitary combinatorial content?