Three Aspects of Gödel's Program: Supercompactness, Forcing axioms, Ω-logic

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Wien

The Π^1_2-statement OCA*

Corollary (Todorčević, 1989)

Assume $P \subseteq \mathbb{R}^2 \setminus \Delta$ is an open symmetric set. Then exactly one of the following holds:

1. *There is a closed uncountable set C such that $C^2 \subseteq P$,*
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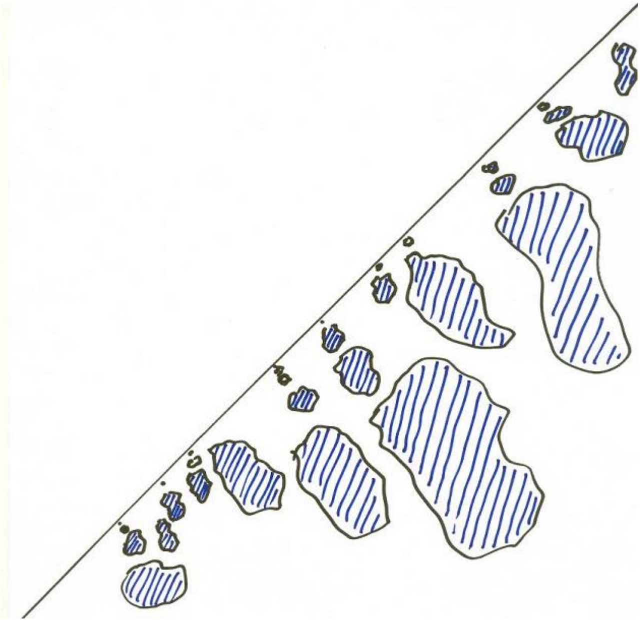
Symmetric sets which do not intersect the diagonal are determined by their intersection with the half plane

$$H = \{(x, y) \in \mathbb{R}^2 : x > y\}$$

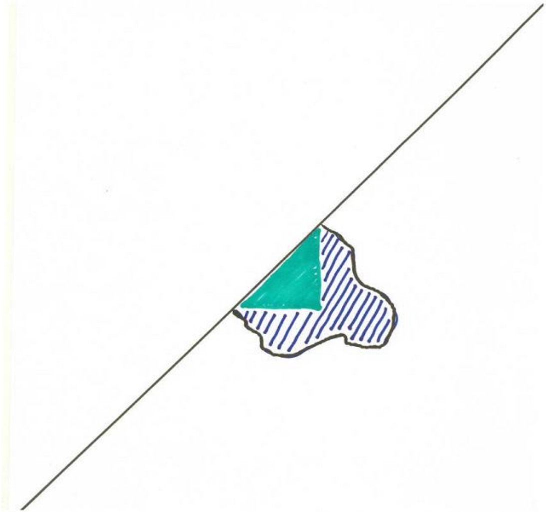
Open partitions of H. I

Open partitions of H. II



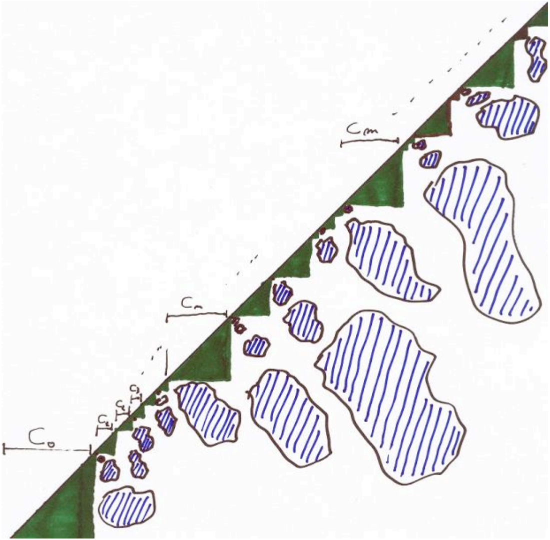
Open partitions of H. III

A trivial example of case 1



Open partitions of H. IV

An example of case 2



Open partitions of H. V

A non trivial example example of case 1

Corollary of what???

Theorem (Todorčević)

Assume the proper forcing axiom PFA.
Then for every $X \subseteq \mathbb{R}$ and every open and simmetryc $P \subseteq \mathbb{R}^2$ exactly one of the following holds:

1. *There is a closed set C such that $C^2 \subseteq P$ and $C \cap X$ is uncountable,*
2. *There is a countable family of closed sets C_n such that $X \subseteq \bigcup_{n \in \mathbb{N}} C_n$ and $C_n^2 \cap P = \emptyset$ for all n.*

Theorem (Shoenfield, 1961)

Assume ϕ is a Π^1_2-statement. If there is an uncountable transitive model M of ZFC *such that $M \models \phi$, then ϕ holds in all transitive uncountable models of* ZFC.

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On the other hand non Π^1_2-problems may show up with more frequency in general topology, functional analysis, homological algebra, category theory...

Theorem (Baumgartner, 1984)

Assume there is a model of ZFC *with a supercompact cardinal. Then there is a model of* ZFC + PFA.

OCA*

For every open and simmetryc partition P of the plane $\mathbb{R}^2$ exactly one of the following holds:

1. There is an uncountable closed set C such that $C^2 \subseteq P$,
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is the Π^1_2-property

$$\forall P \phi(P) \rightarrow (\exists C \psi(P, C) \vee \exists \vec{C} \theta(P, \vec{C}))$$

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- $\theta(P, (C_n^2 : n \in \omega)) \equiv \ldots$

"Platonistic" proof of the Corollary

If we assume a platonistic stand-point and accept large cardinals axioms, the corollary is an immediate consequence of the three theorems since:

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3. By Shoenfield's absoluteness:
   If OCA* holds in some transitive uncountable model M of ZFC, then it holds in all uncountable transitive models M of ZFC.

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3. By Shoenfield's absoluteness:
   If OCA^* holds in some transitive uncountable model M of ZFC, then it holds in all uncountable transitive models M of ZFC.

Thus OCA^* is true.
A posteriori an "ordinary" proof of OCA^* has been found.

1. Forcing axioms.

2. Ω-logic and absoluteness

3. Large cardinals and forcing axioms

Forcing axioms solve problems!

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Take a mathematical problem which is likely to be independent of ZFC, then there are great hopes that PFA will decide it.

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Theorem (Todorčević-Veličković (1992), many others and many proofs afterwards)

PFA $\rightarrow 2^{\aleph_0} = \aleph_2$.

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Theorem (V. (2006))

PFA $\rightarrow$ SCH

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Theorem (Solovay-Tennenbaum (1971))

PFA $\rightarrow$ SH *(In fact* MA $\rightarrow$ SH*).*

The S-space problem:

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Theorem (Moore (2006))

Yes, there is.

The five element basis for the uncountable linear orders:

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Theorem (Moore (2006), culminating the work of Baumgartner, Shelah, Todorčević and others)

Assume PFA. *Then there are five uncountable linear orders such that any other uncontable linear order contains an isomorphic copy of one of these five.*

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Theorem (Farah, 2011, culminating researches by himself, Shelah, Veličković and many others)

Assume PFA*. Then all automorphisms of the Calkin algebra are inner.*

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Truth in $V^{\mathbb{B}}$ is "computable" and depends from the combinatorial properties of $\mathbb{B}$ and from the first order theory of V.

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Definition

$V \models_\Omega \phi$ iff $V^{\mathbb{B}} \models \phi$ for all complete Boolean algebras $\mathbb{B} \in V$.

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Theorem (Woodin, late eighties (in print 1999))

Assume V is a transitive model of ZFC*. *Then for all complete Boolean algebras $\mathbb{B} \in V$ and all statements ϕ:*

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If one is eager to accept large cardinal axioms as true, Ω-truth is absolute with respect to the forcing method.

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Theorem (Woodin, unpublished)

Assume ϕ *is a mathematical statement such that*

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In the presence of large cardinals any problem which can be formulated in the theory of $L(\mathbb{R})$ or even in the theory of $L(P_{\omega_1}\mathrm{Ord})$ cannot be shown independent with respect to ZFC^* using forcing.

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It is well known that any mathematical problem which is expressible as a Π^1_n-property has very high chances to be settled by AD.
For example:

Theorem

Assume AD. *Then every set of reals has the Baire property and is either countable or contains a closed uncountable set.*

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We also need good criteria to accept them.

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- *Can we effectively compute the theory of* $L(P_{\omega_2}\,\mathrm{Ord})$*?*
- *For example: does* $L(P_{\omega_2}\,\mathrm{Ord}) \models \mathrm{AC}$*?*
- *Is the theory of* $L(P_{\omega_2}\,\mathrm{Ord}))$ *invariant with respect to* Ω*-logic?.*

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Assume PFA *holds in a transitive model V. Is there a transitive inner model of V with a supercompact cardinal?*

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It is plausible to conjecture that

> There is a "canonical" inner model for a supercompact cardinal if and only if such a canonical model can be built assuming PFA.

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If it were possible to show that inner models for large cardinals are simply definable assuming strong forcing axioms this would give more ground to accept them as a reasonable strengthening of the notion of large cardinal or even as "generic large cardinals axioms".

THANK YOU FOR YOUR ATTENTION