Comparing the complexity of unstable theories

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Cantor's proof

- All algebraic numbers are roots of polynomials with rational coefficients.
- There are only countably many such polynomials.
- The reals are uncountable.
- Thus transcendental numbers exist.

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Model theory

How many consistent 1-types using countably many parameters?
ACF_0 – countably many versus $(\mathbb{Q}, <)$ – uncountably many

- Morley 1965: theories stable in every cardinal (ω-stable)
- Shelah 1978: theories stable in *some* cardinal (stable)
- Unstable theories

Model theory as the "geography of mathematics"

Definition

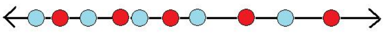
T is λ-unstable if $\exists\ M \models T$, $|M| = \lambda$ and $|S_1(M)| > \lambda$.
T is unstable if λ-unstable in every λ

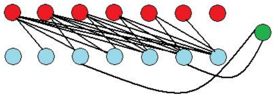
Example

- $(\mathbb{Q}, <)$; 1-types are cuts
- Random graph; 1-types are partitions

Shelah's characterization

T unstable iff some φ has *order property*
i.e. there are $\langle a_i, b_i : i < \omega\rangle$ such that $\models \varphi(a_i, b_j)$ iff $i < j$.





A framework for comparing countable theories

Keisler's order, 1967

$T_1 \trianglelefteq T_2$ if regular ultrapowers of T_1 more likely to be saturated than regular ultrapowers of T_2.

- Preorder (or partial order on $\trianglelefteq$-equivalence classes)
- Compare any two countable theories

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Morley: "The exciting fact is that $\trianglelefteq$ gives a rough measure of the 'complexity' of a theory. For instance, first order number theory is maximal while theories categorical in uncountable powers are minimal."

Keisler's order: formal version

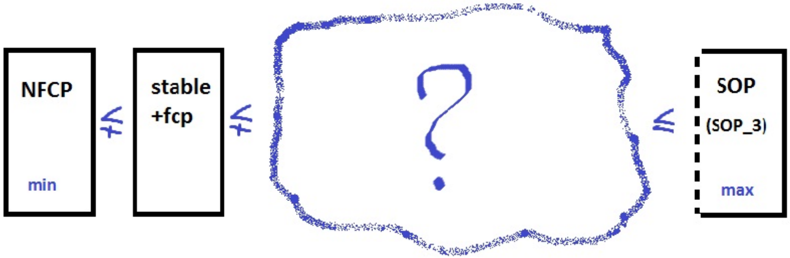
Keisler's order

Let T_1, T_2 be countable theories. Then $T_1 \trianglelefteq T_2$ if for any infinite λ, regular ultrafilter $\mathcal{D}$ on λ, $M_1 \models T_1$, $M_2 \models T_2$, we have: $(M_2)^\lambda/\mathcal{D}$ is λ^+-saturated $\implies$ $(M_1)^\lambda/\mathcal{D}$ is λ^+-saturated.

1. The filter $\mathcal{D}$ on $I = \lambda$ is regular if there is $\{X_i : i < \lambda\} \subset \mathcal{D}$ such that for each $t \in I$, $|\{i : t \in X_i\}| < \aleph_0$.

2. Key property: Level of saturation depends only on $\mathcal{D}$ and T
If $\mathcal{D}$ is a regular ultrafilter on λ and $M \equiv N$ in a countable language then $M^\lambda/\mathcal{D}$ is λ^+-saturated iff $N^\lambda/\mathcal{D}$ is.

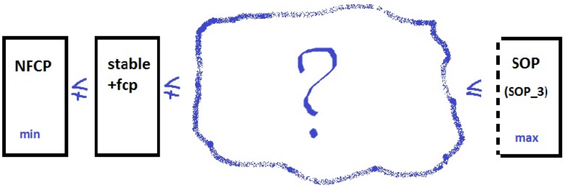
Results on Keisler's order, 1967-1978



Shelah, Keisler, Džamonja, Usvyatsov...

1. Minimum class: not the finite cover property
   [Algebraically closed fields...]
2. Class 2: All remaining stable theories
   [Equivalence relation with a class of size n for each n...]
3. Maximum class: definable linear order is sufficient,
   1996-2008: SOP_3 sufficient, necessity open
   [$(\mathbb{Q}, <)$, real closed fields, first order number theory...]

The unstable case: Localization



The two known classes depend on properties of formulas; this will always be true.

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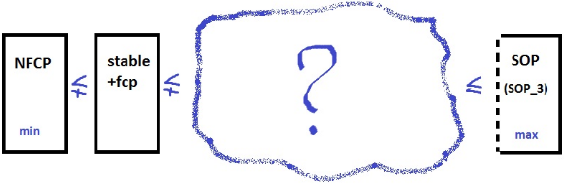


The two known classes depend on properties of formulas; this will always be true.

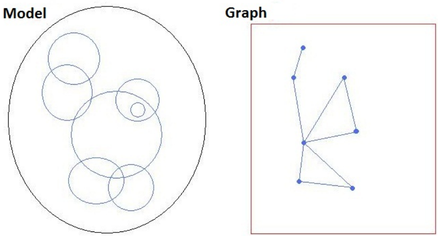
Localization Theorem (Malliaris 2009 [2])

Let $\mathcal{D}$ be a regular ultrafilter on $\lambda \geq \aleph_0$ and let T be a countable theory, $M \models T$. Then $M^\lambda/\mathcal{D}$ is λ^+-saturated iff, for all formulas φ, $M^\lambda/\mathcal{D}$ realizes all φ-types over sets of size $\leq \lambda$.

Program: Keisler's order rests on suitable analysis of formulas.

Formula complexity via characteristic sequence

Associate to (T, φ) a characteristic sequence $\langle P_n : n < \omega \rangle$



Left: $\varphi(x; a)$, $\varphi(x; b)$.... *Right: parameters* a, b...

For each n, let $P_n(z_1, \ldots z_n) := \exists x \bigwedge_{i \leq n} \varphi(x; z_i)$

Consider $p(x) = \{\varphi(x; a) : a \in A\}$. P_∞ = for all P_n

- p is a consistent partial type iff A is a P_∞-complete graph
- p is k-inconsistent iff A is a P_k-empty graph

The exploration begins, [3], [4]

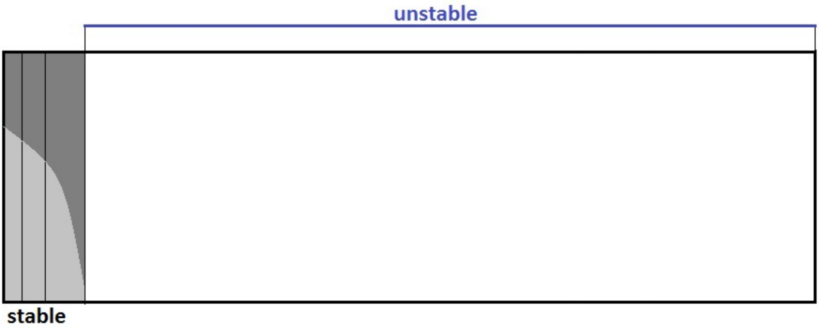
A type in an ultrapower $M^\lambda/\mathcal{D}$ is a P_∞-complete graph of size λ. It is realized iff its projections to the index models are a.e. P_∞-complete graphs.

All known Keisler classes appear naturally in this framework.

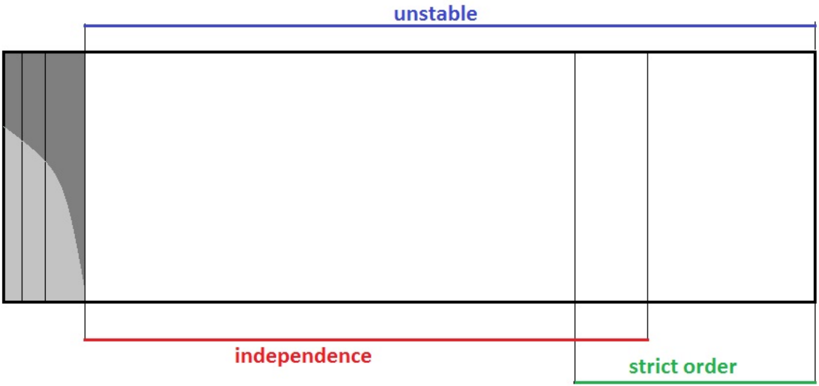
Persistence Theorems, Malliaris [3]

- *Stability, simplicity, and NIP (not independence property) can be characterized in terms of persistence of configurations in the characteristic sequence.*
- *φ has the finite cover property just in case the sequence $\langle P_n : n < \omega \rangle$ is not determined by any initial segment.*

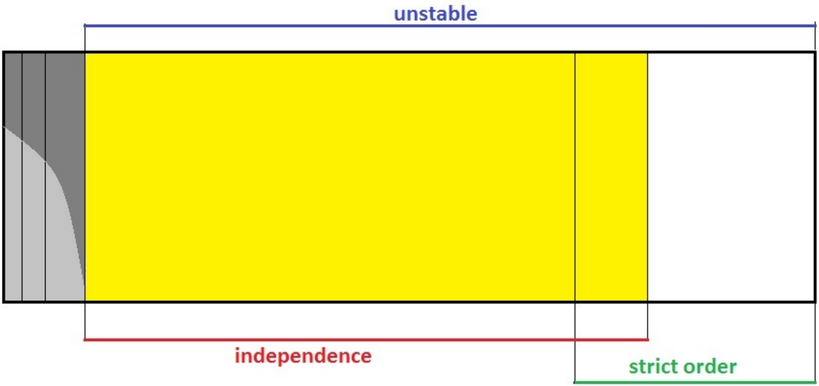
Shelah's classic "dichotomy" for unstable theories



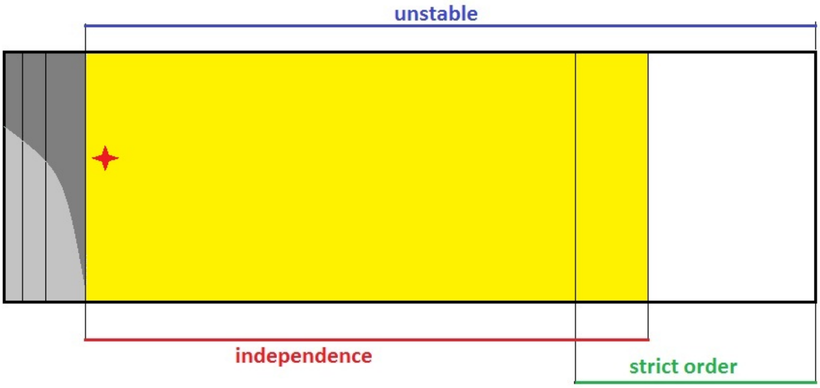
Rigidity/randomness



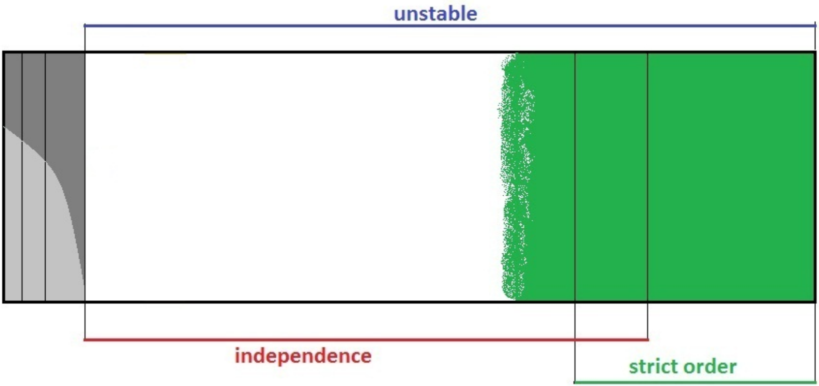
In our context, within the independent theories...



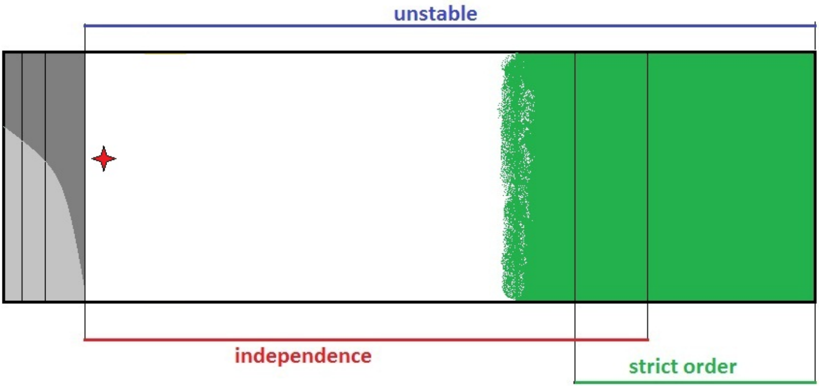
the random graph is $\trianglelefteq$-minimum



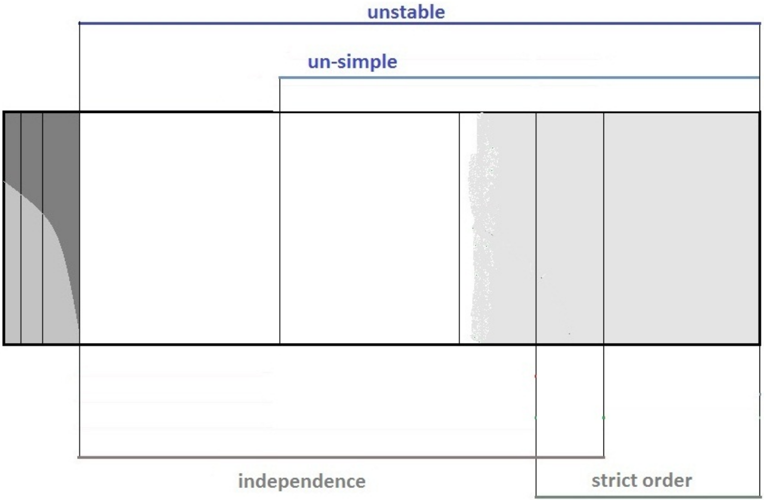
and since strict order is $\trianglelefteq$-maximum



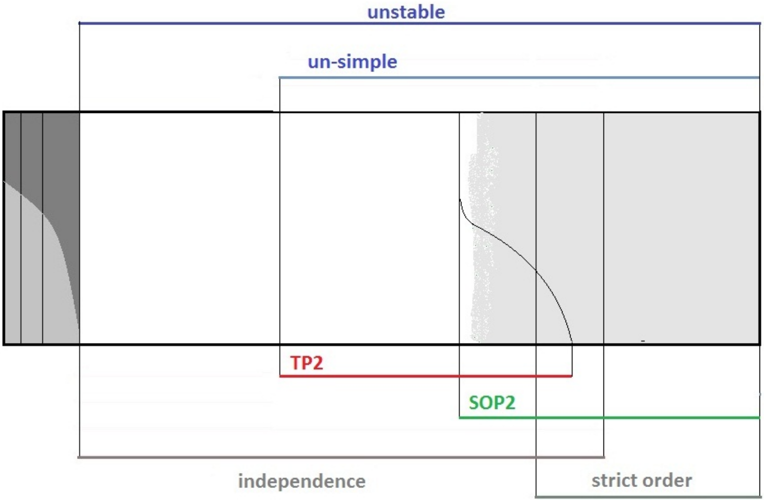
the random graph is minimum among all unstable theories



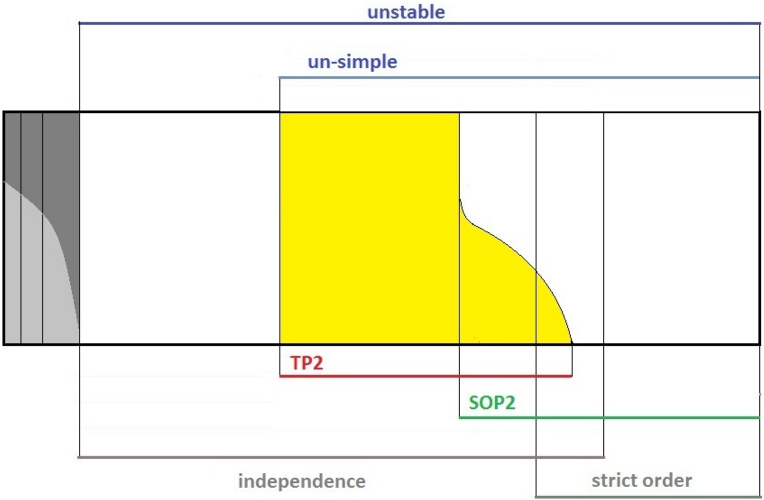
Shelah: a "dichotomy" above simple theories



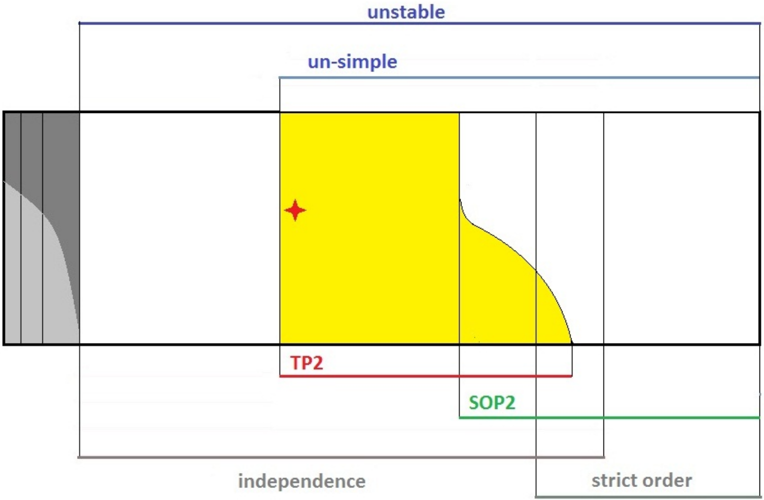
Two kinds of tree property



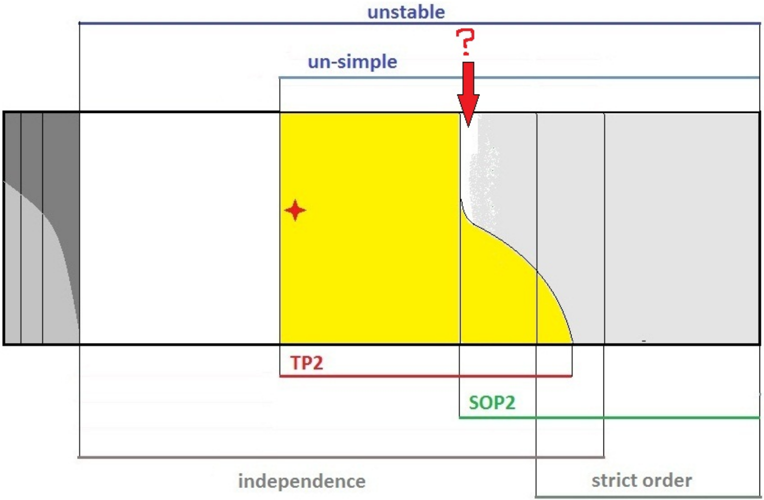
The "independent side" of the dichotomy



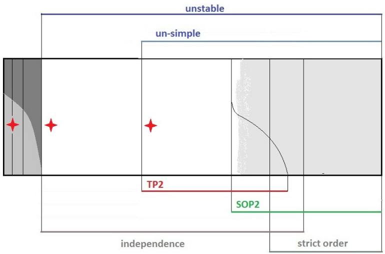
Theorem, Malliaris [5]



Remark



Canonicity of these examples



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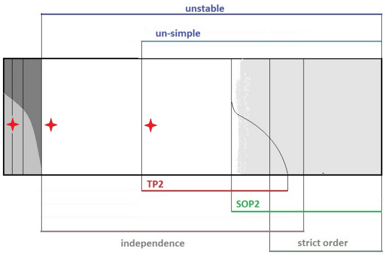
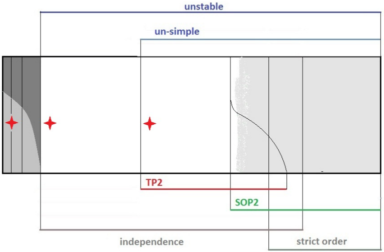


- Described by =-definable characteristic sequences of finite support
- Correspond to partition properties of ultrafilters
- Can one classify all such "fundamental formulas"?

Canonicity of these examples



Theorem (Malliaris 2011 [6])

All fundamental formulas are $\trianglelefteq$-dominated by either: the empty theory, the random graph or the minimum TP_2 theory.

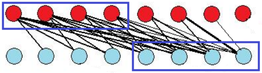
The proof applies Shelah's classification of second-order quantifiers.

Graph theory: a new language suggests new structure

Szemerédi's regularity lemma

Given $\epsilon > 0$, any sufficiently large finite graph can be equitably partitioned into no more than $k = k(\epsilon)$ pieces so that all but ϵk^2 pairs are ϵ-regular.

(X, Y) is ϵ-regular: a measure of uniformity



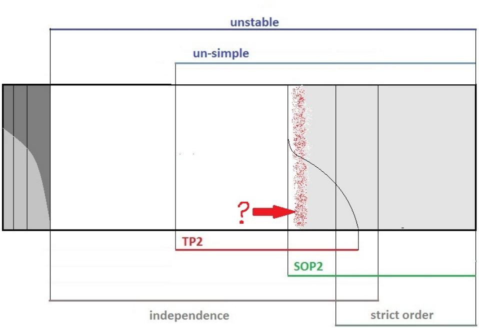
- Density: $d(X, Y) = e(X, Y)/|X||Y|$
- ϵ-regular: For all $|X'| \geq \epsilon|X|$, $|Y'| \geq \epsilon|Y|$, have $|d(X, Y) - d(X', Y')| < \epsilon$.

Applications to characteristic sequence: Malliaris [4]-[5]
Model-theoretic applications to regularity: Malliaris and Shelah [7]

Theorem (Malliaris 2010 [4])

If T is simple, then after localization, the P_2-density between any ϵ-regular pair of P_∞-complete graphs is close to 0 or 1.

$\approx$ on a definable set containing any fixed P_∞-complete graph



"Independence Theorem" for P_2

"generic position," "essentially inconsistent," finitary

By way of conclusion

The longstanding open question of Keisler's order suggests a classification of theories in terms of complexity which picks up on rich new structure, yet correlates with known dividing lines in crucial ways.

Localizing, we build a new language and lay the groundwork for a large-scale analysis of unstable theories: asymptotic behavior visible in characteristic sequences of hypergraphs.

- Illuminates fundamental structure
- Interaction of model theory, graph theory and combinatorics
- Fine structure of interplay between independence and order
- Role of edge distribution and density, complementing the classical analysis
- Calibrate these tools by means of saturation of ultrapowers

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