Dictionary Learning for Positive Definite Matrices

(with Application to Nearest Neighbor Retrieval)

Suvrit Sra[1] and Anoop Cherian[2]

[1] Max Planck Institute for Intelligent Systems, Tübingen
[2] University of Minnesota, MN

Linear Model

$$s \approx Dc$$

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Signal s: Image, audio, video, matrix, tensor

Dictionary D: Basis for linear coding

Code c: code vector, used in application

Linear Model

$$s \approx Dc$$

- Image compression
- Image denoising
- Texture synthesis
- Audio processing
- Signal classification
- Database search and retrieval

Dictionary learning task

Given examples $s_1, \ldots, s_m$, "learn" dictionary D so that $s_j \approx Dc_j$, for suitable c_j

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Data dependent encoding

Versatile applications

Generalized Dictionary Learning

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Generalized Dictionary Learning (GDL)

$$S \approx \mathcal{D} C$$

Matrix S: input data

Dictionary $\mathcal{D}$: tensor

Code C: code matrix

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Generalized Dictionary Learning (GDL)

$$S \approx \mathcal{D}C$$

$$S \approx DCE^T$$

$$\mathcal{D} := (E \otimes D)$$

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	Vectorized	GDL
Storage	$(m+n)d^2 + mn$	$md^2 + 2dn + mn$
Cost / Iter	$O(d^2 n)$	$O(dn)$
Locality	Lost	Preserved

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Let us look at special case of GDL now

GDL for covariance matrices

Input: a set $\{S_1, \ldots, S_m\}$ of covariance matrices

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Find coding matrices $C_1, \ldots, C_m \succeq 0$, and dictionary matrix D

$$\min \tfrac{1}{2} \sum_{i=1}^{m} \|S_i - DC_iD^T\|_F^2 + \sum_{i=1}^{m} \beta_i \mathrm{sp}(C_i),$$

Note: If $C \succeq 0$, then $DCD^T \succeq 0$

Simplification

$$S_i \approx \sum_{j=1}^{n} c_j d_j d_j^T$$

$$\min \tfrac{1}{2} \sum_{i=1}^{m} \|S_i - DC_iD^T\|_F^2 + \sum_{i=1}^{m} \beta_i \|C_i\|_1$$

$$S_i \approx DC_iD^T$$

C_i is nonnegative, diagonal

GDL - how to optimize

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  1. Initialize dictionary
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  5. Repeat

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Coding: Solve large-scale NNLS
Dict Update: Projected stoch. gradient step

Coding covariance matrices

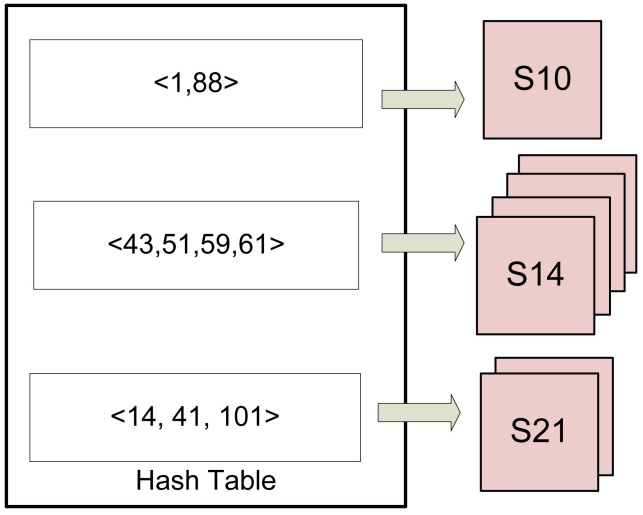
- Dict: n elements; say r sparse code a covariance
- $\binom{n}{r}$ possible sparse codes – huge space of codes
- High chance that codes provide unique hash keys
- Hashing facilitates fast nearest neighbor retrieval

Subspace Combination Tuple

- Covariance S, and dictionary $D \in \mathbb{R}^{d \times n}$
- Let each element $d_i \in D$ have unique integer identifier u_i
- Suppose $d_i, d_j, \cdots, d_k$ used when sparse coding S
- *Subspace Combination Tuple* (SCT): Sort$(u_i, u_j, \cdots, u_k)$

Hashing

- Encode S into its SCT
- Use SCT to index into a hash table
- Collisions resolved using search
- Allows fast NN search for given input covariance



Three main experiments

1. Performance on real-world noisy data (LabelMe images)
2. Larger covariances (40×40 from Face image dataset)
3. Larger dataset (40,000 covariances of textures)

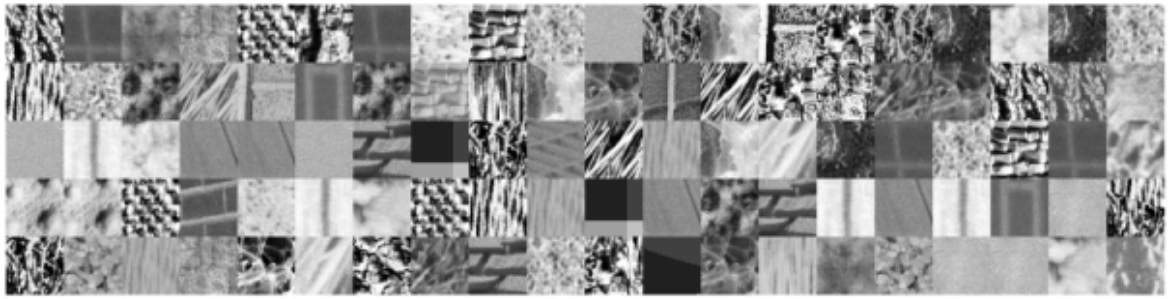
Experiments - LabelMe object dataset

- Took 10,000 annotated images dataset.
- For each annotated blob in an image: extracted color RGB values; first and second order gradients.
- Used these to compute the covariance matrix
- Obtained 25,000 such covariances (7×7)

Experiments - FERET face pose dataset

- Dataset has facial appearances, with varying poses.
- Aim: recognize person, irrespective of pose.
- We took approx. 10,000 images
- Computed Gabor transform based 40 $\times$ 40 covariances (as per state-of-the-art)

Experiments - Texture Classification



- Texture key in satellite imaging, inspection systems, etc.
- Brodatz (111 classes) and Curret texture (60 classes)
- Covariances from random patches of each class.
- Pixel coordinates, intensity, first-order gradients to create covariances (5 x 5)

Dictionary Learning - Setup

- Cross-validation to estimate dictionary size
- Subset chosen as database; another fraction as query set
- GDL in MATLAB; while L2LSH, VEC in C
- Riemannian metric used to measure “nearness”

Evaluation

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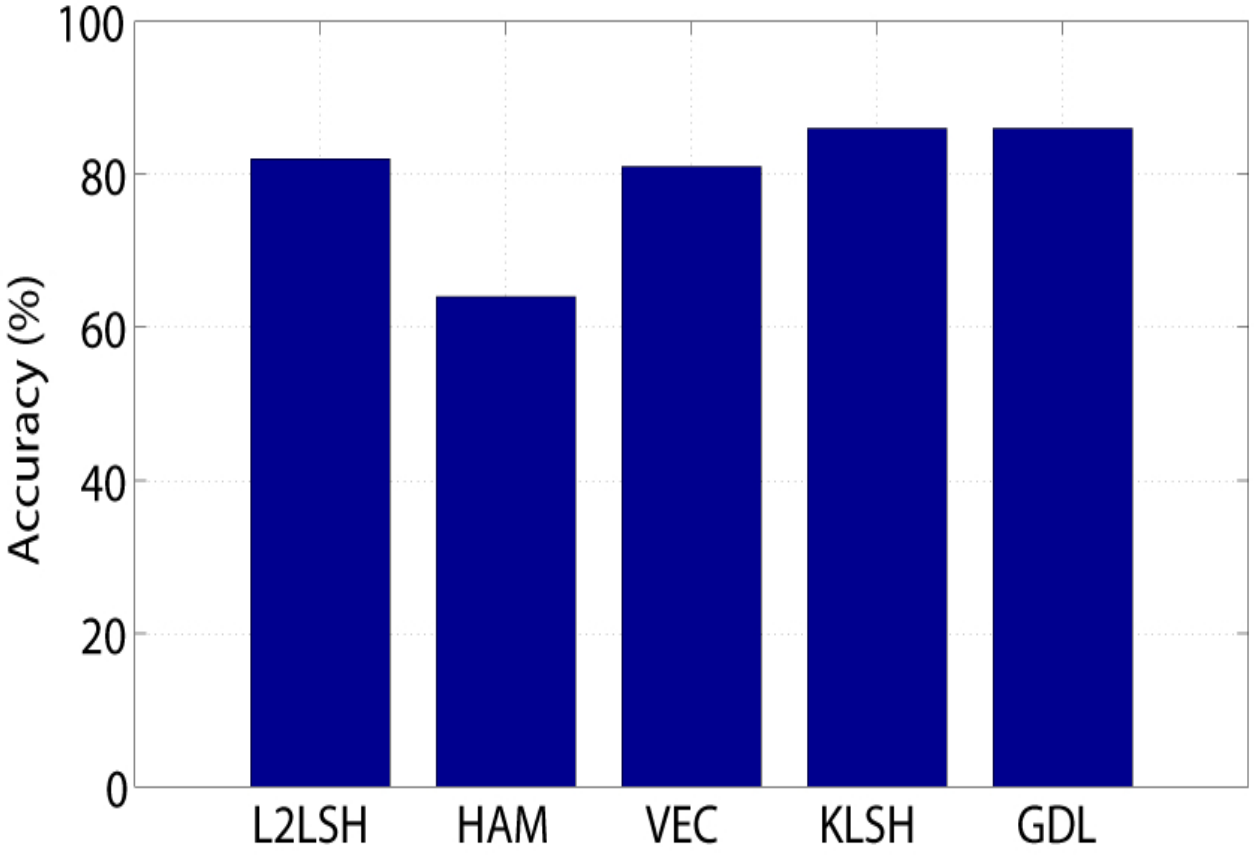
- We chose $\epsilon = 0.75$
- Using this ANN setup, we define *Accuracy* as:

$$\text{Accuracy} := \frac{\#\text{correct matches}}{\#\text{queries}}.$$

Competing Methods

1. Log-Euclidean Embedding
   - Embed into euclidean space using matrix logarithm
   - Locality sensitive hashing (L2LSH, Hamming) based NN
2. Vectorization (VEC)
   Vectorize without embedding, and use LSH
3. Kernelized LSH
   Use pseudo-kernel function based on geodesic distance.

Results



NN retrieval accuracy: faces dataset

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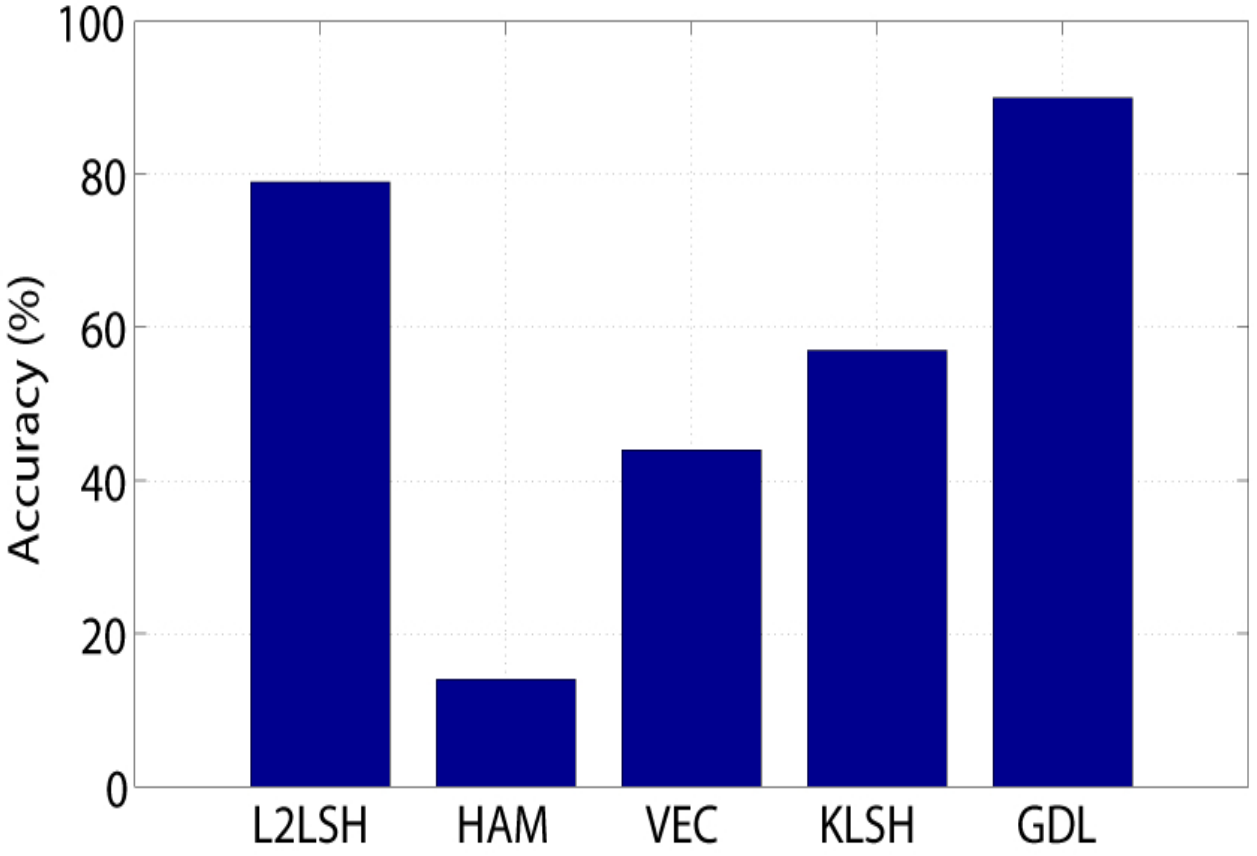
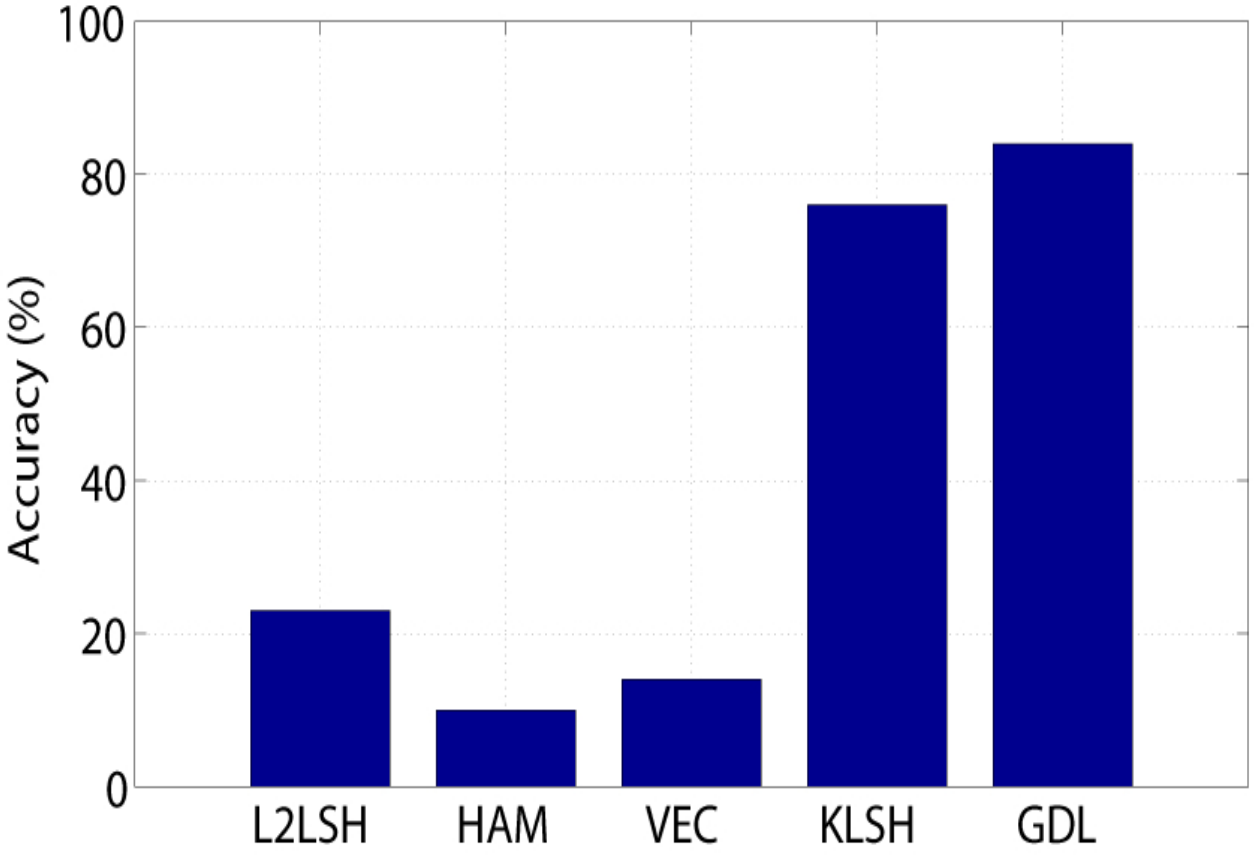


NN retrieval accuracy: texture dataset

Results



NN retrieval accuracy: Objects dataset

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