Sparse Estimation with Structured Dictionaries

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Sparse estimation problem:

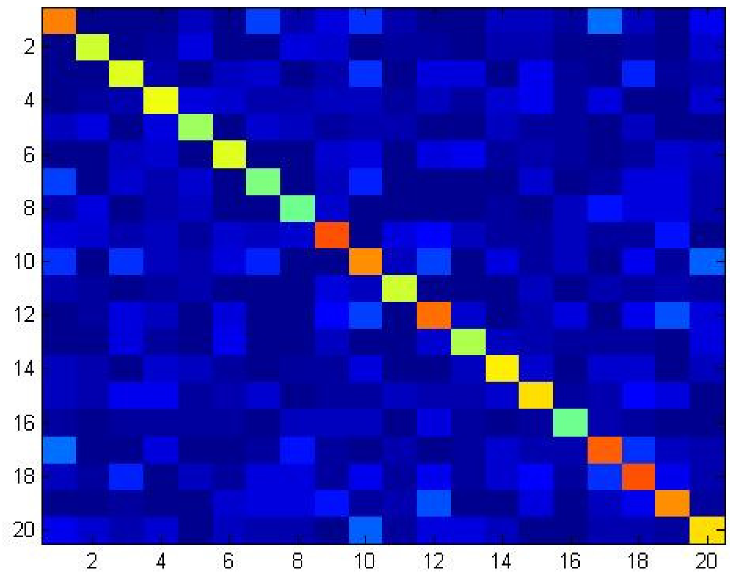
$$\min_{\mathbf{x}} \quad \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{y} = \underbrace{\Phi}_{\text{overcomplete dictionary of basis vectors}}\mathbf{x}$$

- Non-convex, combinatorial problem in general.
- Convex relaxation using the ℓ_1 norm produces an equivalent solution if Φ is sufficiently *unstructured*.

Dictionary Correlation Structure

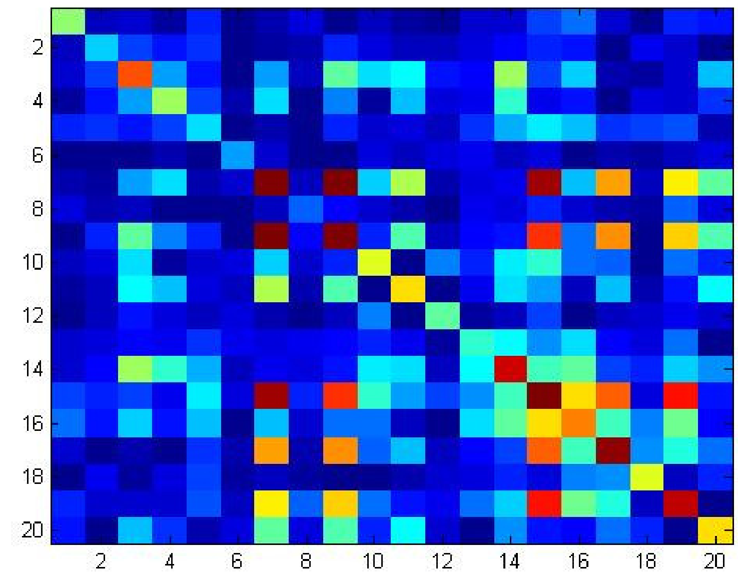
Unstructured

$$\Phi^T \Phi$$



Structured

$$\Phi^T \Phi$$



Examples:

$$\Phi_{(unstr)} \sim \text{iid } N(0,1) \text{ entries}$$

$$\Phi_{(unstr)} \sim \text{random rows of DFT}$$

Example:

$$\Phi_{(str)} = \underbrace{\mathrm{W}}_{\text{arbitrary}} \cdot \Phi_{(unstr)} \cdot \underbrace{\mathrm{D}}_{\text{block diagonal}}$$

New Strategy

- Apply a Φ-dependent projection that maps $\mathbf{x}$ to a new space

$$\mathbf{z} = \mathrm{P}_{\Phi}(\mathbf{x})$$

- Use a standard sparsity penalty g in this new space and solve:

$$\min_{\mathbf{x}} \sum_i g(z_i) \quad \text{s.t.} \quad \begin{array}{l} \mathbf{y} = \Phi \mathbf{x} \\ \mathbf{z} = \mathrm{P}_{\Phi}(\mathbf{x}) \end{array}$$

The projection operator:

1. Must compensate for dictionary structure.
2. Preserve sparsity, meaning if $\mathbf{z}$ is maximally sparse, $\mathbf{x}$ is also maximally sparse.

Analysis

- Convenient optimization via reweighted ℓ_1 minimization
- Provable performance improvement in certain situations

- **Toy Example:**
- Generate 50-by-100 dictionaries:

$$\Phi_{(\text{unstr})} \sim N(0,1), \quad \Phi_{(\text{str})} = \Phi_{(\text{unstr})} \cdot D$$

- Generate a sparse $\mathbf{x}$
- Estimate $\mathbf{x}$ from observations

$$\mathbf{y}_{(\text{unstr})} = \Phi_{(\text{unstr})} \cdot \mathbf{x}, \qquad \mathbf{y}_{(\text{str})} = \Phi_{(\text{str})} \cdot \mathbf{x}$$