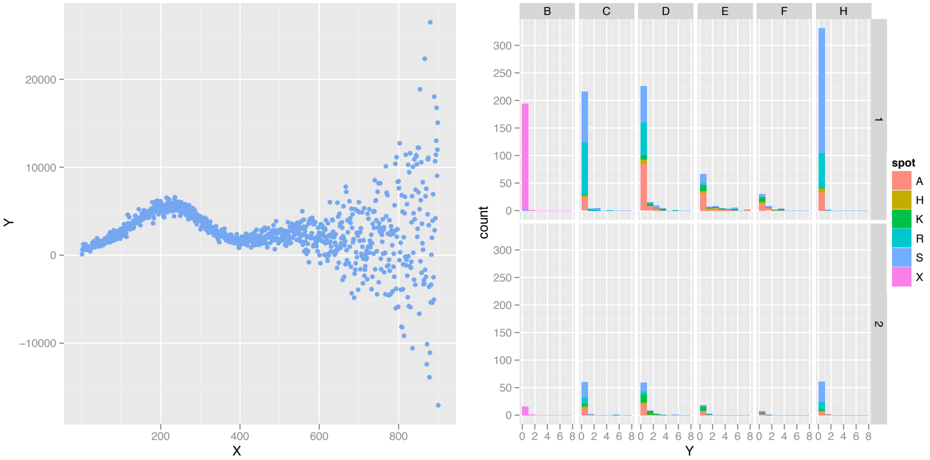
Dirichlet Process Mixtures of Generalized Linear Models

Lauren Hannah[1]
David Blei[2]
Warren Powell[1]

Princeton University
[1]Department of Operations Research and Financial Engineering
[2]Department of Computer Science

May 13, 2010

Nonparametric Regression



Nonparametric Regression

- Covariates X and response Y
- X and Y may have different forms (continuous, count, categorical)
- Goal: prediction, ie compute $\mathbb{E}[Y|X = x]$
- Parametric regression restricts shape (a straight line, polynomial, etc)
- Nonparametric regression tries to fit a function

Motivation

Nonparametric Regression Goals

- Flexible model
- Accommodate input/output types
- Be successfully applied to data with different characteristics
- Theoretical assurances, like asymptotic unbiasedness
- Computational tractability

Dirichlet Process Mixtures of Generalized Linear Models

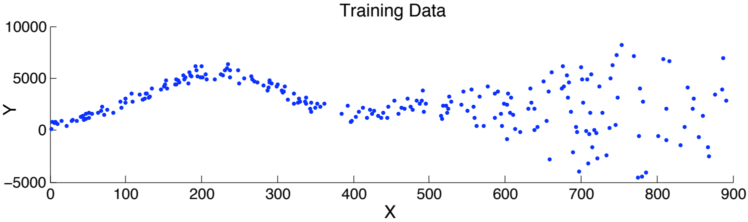
Idea!

- *Locally*, a complex model can be represented by a simpler model
- Dirichlet process mixture models:
  - Cluster observations probabilistically
  - Can accommodate many data types
- Cluster data so that a GLM fits well in each cluster
  - Clusters *and* local GLM parameters are latent variables
  - Predict mean response by averaging posterior draws

What am I going to talk about?

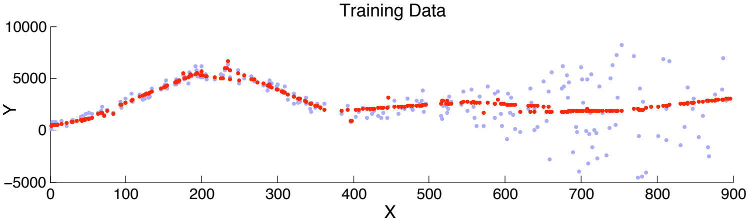
- Abbreviation: DP-GLM
- General regression method for all input types accommodated by DP and output types accommodated by GLM
- Continuous, categorical, count, circular, etc covariates/response
- Generalization of existing special case methods (eg Shahbaba and Neal (2009))
- We give conditions for asymptotic unbiasedness

DP-GLM: Intuition



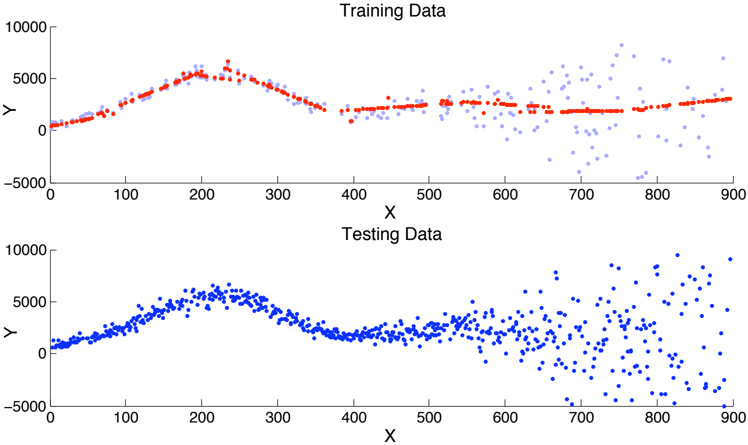
Start with training data.

DP-GLM: Intuition



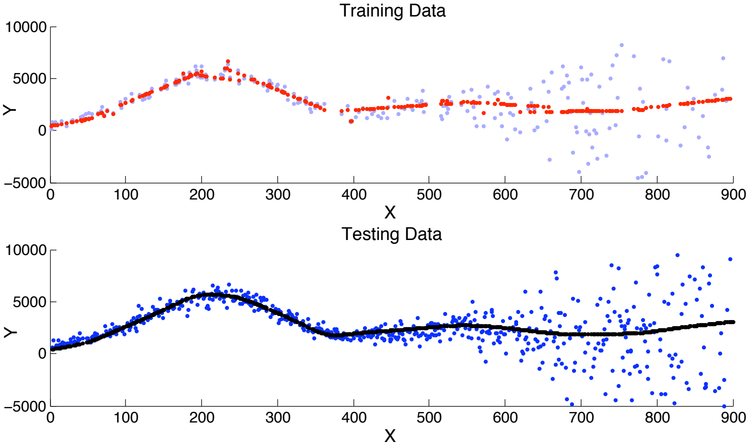
Cluster and fit regression probabilistically.

DP-GLM: Intuition



Observe testing data—we want to predict a mean function.

DP-GLM: Intuition



Fit testing covariates into clustered model; average to get mean function.

Review: The Dirichlet Process

Properties of the Dirichlet Process

- A distribution over distributions–i.e. a draw from a DP is a random measure
- Random measures from DPs are almost surely discrete
  - When used as a distribution on hidden parameters, this produces a clustering effect
- Parameterized by base probability measure G_0 and scale α
- If $\theta_1, \dots, \theta_n \sim P$, $P \sim DP(\alpha G_0)$, then

$$\theta_{n+1} | \theta_{1:n} \sim \frac{1}{\alpha + n} \sum_{i=1}^{n} \delta_{\theta_i} + \frac{\alpha}{\alpha + n} G_0$$

- Use as prior on distribution for hidden parameters θ_i

Dirichlet Process Mixtures of Generalized Linear Models

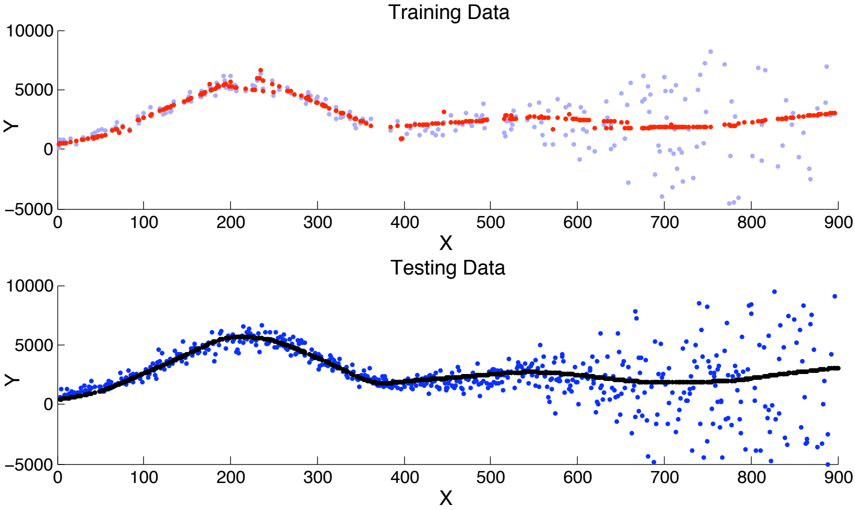
Dirichlet Process Mixtures of Generalized Linear Models (DP-GLM) for covariates X and response Y:

$$
\begin{aligned}
P &\sim DP(\alpha G_0) \\
\theta_i | P &\sim P \\
X_i | \theta_i &\sim f_x(x | \theta_{i,x}) \\
Y_i | \theta_i, X_i &\sim f_y(y | X_i, \theta_{i,y})
\end{aligned}
$$

Example: Gaussian Model: $X, Y \in \mathbb{R}$

$$
\begin{aligned}
P &\sim DP(\alpha G_0) \\
\theta_i = (\mu_{i,x}, \sigma_{i,x}, \beta_{i,0}, \beta_{i,1}, \sigma_{i,y}) | P &\sim P \\
X_i | \mu_{i,x}, \sigma_{i,x} &\sim N(\mu_{i,x}, \sigma^2_{i,x}) \\
Y_i | \beta_{i,0}, \beta_{i,1}, \sigma_{i,y}, X_i &\sim N(\beta_{i,0} + \beta_{i,1} X_i, \sigma^2_{i,y})
\end{aligned}
$$

DP-GLM: Gaussian Model



Computational Procedure

Given data $D = (X_i, Y_i)_{1:n}$, we want to compute $\mathbb{E}[Y|X = x, D]$

1) Choose the GLM
2) Choose DP base measure G_0
3) Estimate posterior of $\theta_{1:n}$ given $(X_i, Y_i)_{1:n}$
    - We use Gibbs sampling, Neal (2000) Algorithms 3, 6 or 8
    - Obtain M i.i.d. samples of $\theta_{1:n}^{(m)}$ from the posterior
4) Compute predicted value $\mathbb{E}[Y|X = x, D]$:

$$\mathbb{E}[Y|X = x] = \mathbb{E}\left[\mathbb{E}[Y|X = x, D, \theta_{1:n}]\right]$$

Computational Procedure

Computing the prediction $\mathbb{E}[Y|X = x, D]$

- Given $\theta_{1:n}$, we can compute expectation:

$$\mathbb{E}[Y|x, \theta_{1:n}] = \frac{1}{b}\sum_{i=1}^{n} \mathbb{E}\left[Y|x, \theta_i\right] f_x(x|\theta_i) + \frac{\alpha}{b}\int \mathbb{E}\left[Y|x, \theta\right] f_x(x|\theta) G_0(d\theta),$$

$$b = \alpha \int f_x(x|\theta) G_0(d\theta) + \sum_{i=1}^{n} f_x(x|\theta_i).$$

- Get M observations of $\theta_{1:n}$
- But $\theta_{1:n}$ is unknown, so we average over samples $(\theta_{1:n}^{(m)})_{m=1}^{M}$

$$\mathbb{E}\left[Y|X = x, D\right] \approx \sum_{m=1}^{M} \mathbb{E}\left[Y|X = x, D, \theta^{(m)}\right]$$

Theoretical Properties of the DP-GLM

Asymptotic Unbiasedness

- Want our estimate of the mean function to converge to the true mean function as we get more observations
- This is not a given with Dirichlet process priors (Diaconis and Freedman, 1986)
- Asymptotic unbiasedness depends on:
  - True distribution of X, Y, denoted $f_0(x, y)$
  - Model (i.e. DP-GLM parametric functions)
  - Base measure G_0

Theoretical Properties of the DP-GLM

Theorem

The DP-GLM is asymptotically unbiased in a compact set of covariates $\mathcal{C}$ if:

(i) *(K-L Condition) for every $\delta > 0$, prior puts positive measure on*

$$\left\{ f : \int f_0(x,y) \log \frac{f_0(x,y)}{f(x,y)} dxdy < \delta, \right.$$
$$\left. \int f_0(x,y) \left(\log \frac{f_0(x,y)}{f(x,y)} \right)^2 dxdy < \delta \right\},$$

(ii) $\int |y|^2 f_0(y|x) dy < \infty$ *for every $x \in \mathcal{C}$, and*

(iii) *there exists an $\epsilon > 0$ such that for every $x \in \mathcal{C}$,*

$$\int \int |y|^{1+\epsilon} f_y(y|x,\theta) G_0(d\theta) < \infty.$$

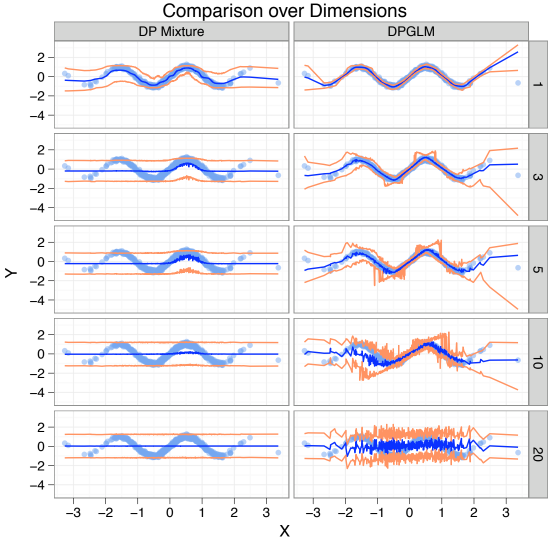
Theoretical Properties of the DP-GLM

Satisfying Main Theorem

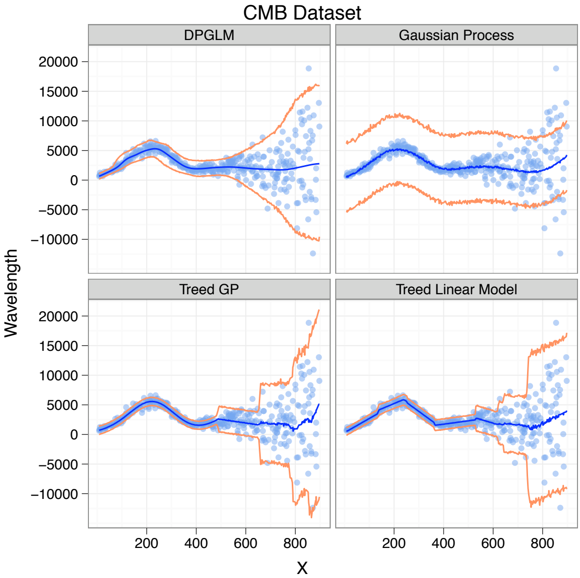
- K-L condition is hard to show.
- When is it satisfied?
  - Gaussian Model: conjugate base measures, shown in slide.
  - Continuous and categorical covariates/response can be used as well with conjugate base measures.
- The rest is an open question.

Empirical Analysis

DP-GLM Comparison: DP regression *without* GLM



DP-GLM Comparison: Heteroscedastic Data



Cosmic Microwave Background (CMB) Bennett et al. (2003)

- Power spectrum vs. multipole moments.
- One continuous covariate, continuous response.
- Heteroscedastic noise.

Concrete Compressive Strength (CCS) Yeh (1998)

- Concrete compressive strength against composition covariates (cement, water, fly ash, etc).
- Eight continuous covariates, one continuous response.
- Low noise, moderate dimensionality.

Solar Flare (Solar) Bradshaw (1989)

- Number of solar flares vs. sun features (solar spots, etc).
- Eleven categorical covariates, count response.
- Moderate dimensionality, atypical covariate/response types.

Competitors

Competitors

- Least squares linear regression (for CMB, CCS)
- Tree regression (CART), treed linear models
- Gaussian process prior regression, treed Gaussian processes
- Dirichlet process regression *without* GLM
- Poisson regression (for Solar)

Numerical Results: Cosmic Microwave Background

Covariates:

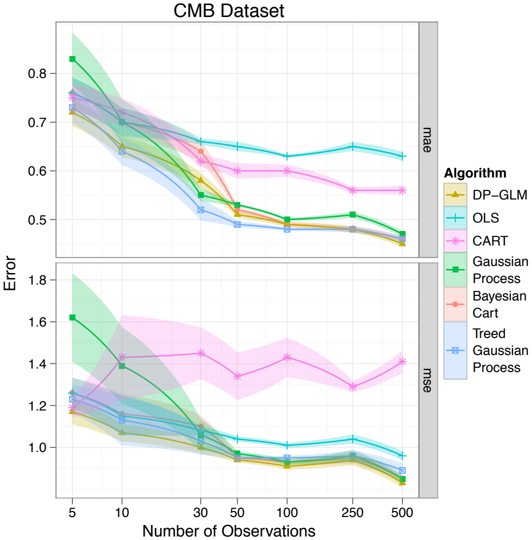
- 1 continuous

Response:

- continuous

Other:

- heteroscedastic



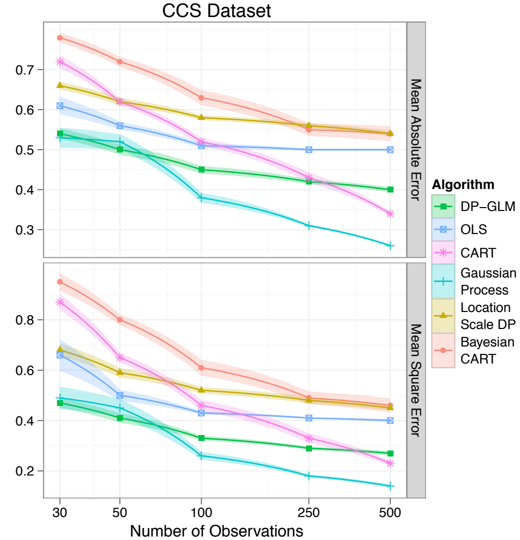
Numerical Results: Concrete Compressive Strength

Covariates:

- 8 continuous

Response:

- continuous



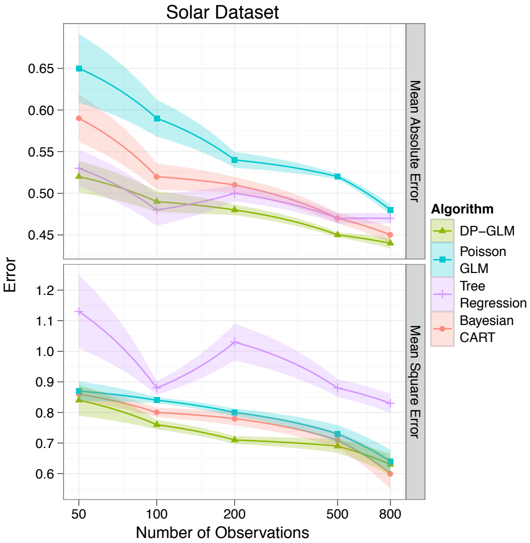
Numerical Results: Solar Flare

Covariates:

- 11 categorical

Response:

- count



Summary

DP-GLM Issues/Future Work:

- Automate choice of G_0, hyperparameters
- Investigate balance between modeling covariates and response

DP-GLM Pros:

- Flexible nonparametric regression method; can be used in many settings
- Generally competitive with state of the art regression methods
- Generally stable outputs
- Can accommodate heteroscedasticity, overdispersion in a natural manner

Thank You!

Lauren Hannah
lhannah@princeton.edu