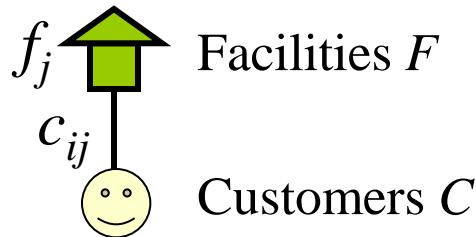


# Solving the Facility Location Problem Using Message Passing

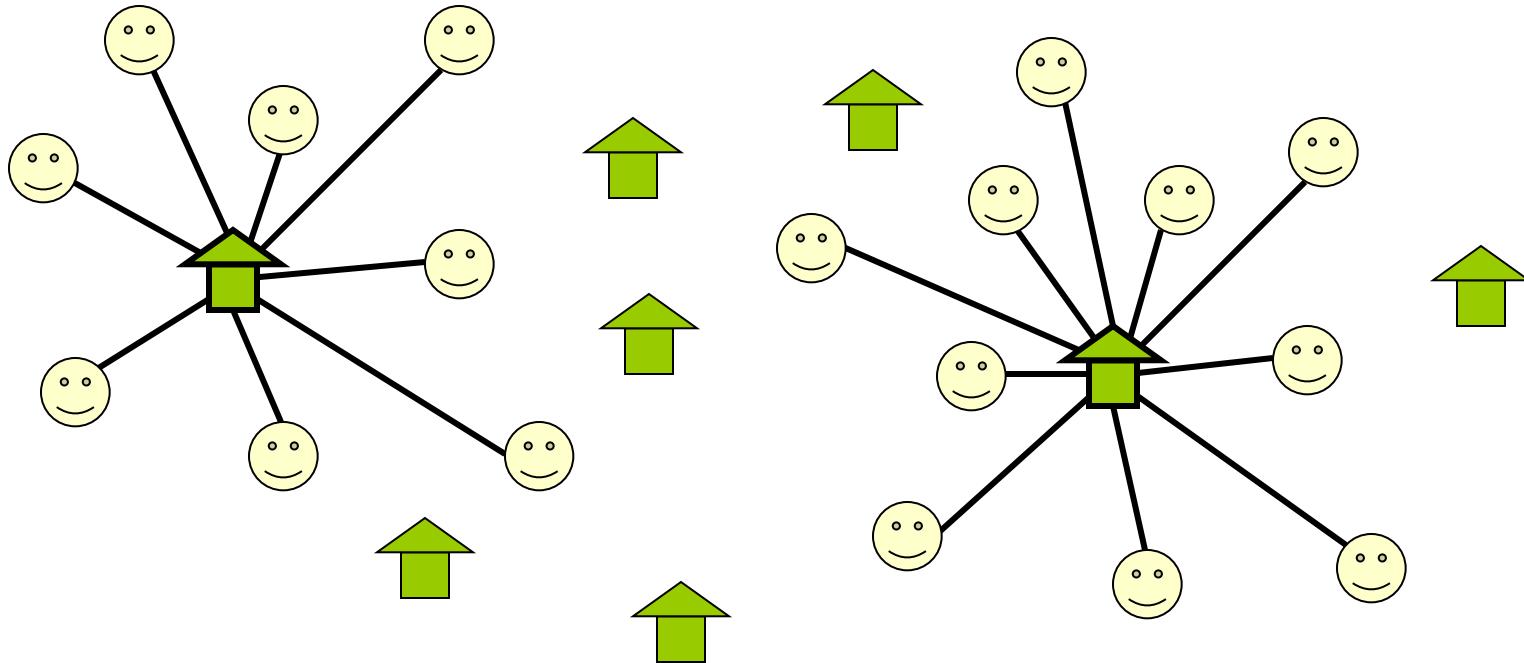


Nevena Lazic, Brendan Frey, Parham Aarabi  
University of Toronto

# Facility Location



**Facility Location (FL) Problem:**  
Open a subset of facilities & connect  
customers to one facility each at minimal cost

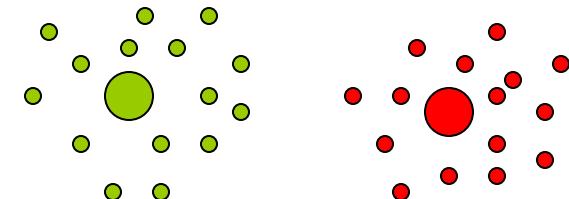


# FL in Machine Learning

---

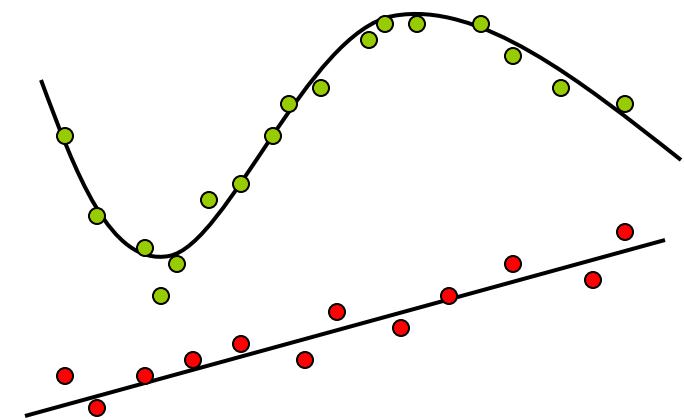
- ❑ Exemplar-based clustering

- $C=F$



- ❑ Multiple model selection [Li`07]

- $F$ : models
  - $C$ : data



- ❑ Many practical problems...

- wireless sensor networks
  - computational biology
  - computer vision

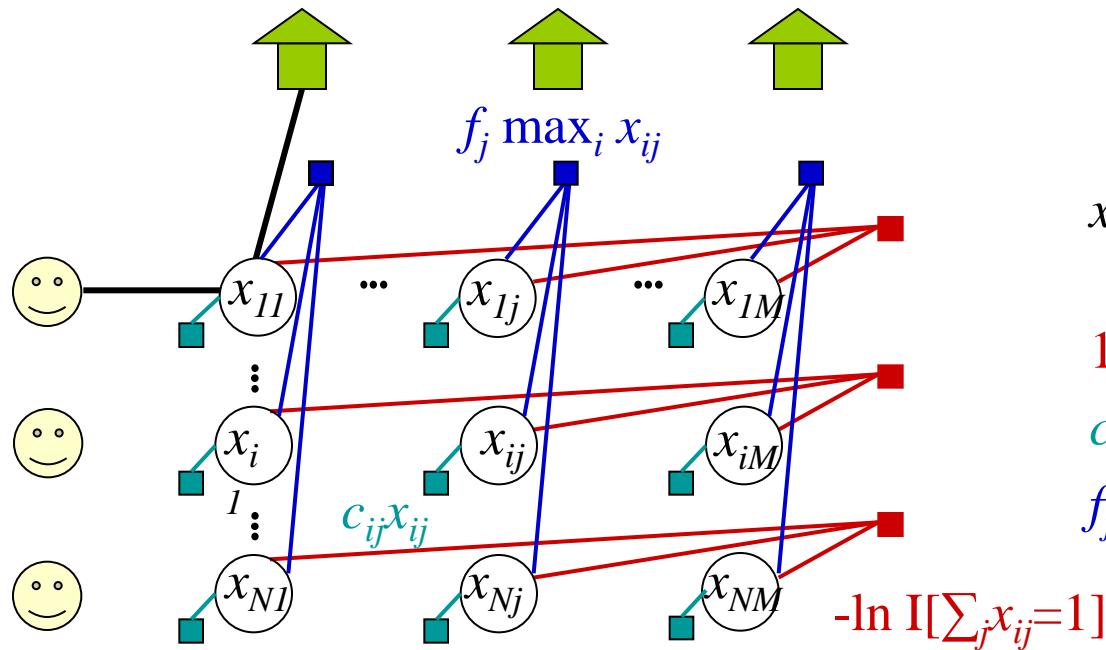
# Outline

---

- Approach to FL:
  - Approximate MAP inference in graphical model
  - Max-product linear programming (MPLP)  
[Globerson & Jaakkola '08]
- MPLP fixed points
  - Unique solution: guaranteed optimal
  - Non-unique solution: unknown how to set some variables
- Today
  - New greedy algorithm for decoding variables for FL
  - In some cases, does not coincide with any MPLP variable assignment
  - Optimality guarantees (3-approximation)
  - Empirically better solutions than typical MPLP solutions

# FL Graphical Model

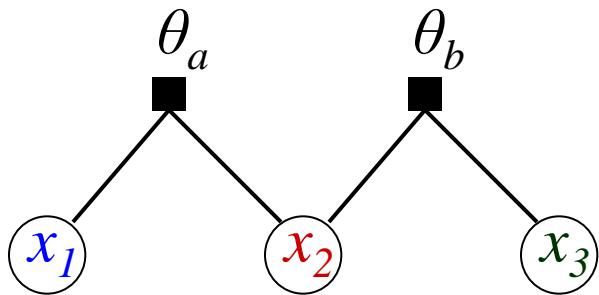
---



$$\begin{aligned}
 & \min_x \quad \sum_{ij} c_{ij} x_{ij} + \sum_j f_j \max_i x_{ij} \\
 \text{s.t.} \quad & \sum_j x_{ij} = 1
 \end{aligned}$$

# Background: MPLP

---



$x_1 x_2$	$\mu_a(x_1, x_2)$	$x_2 x_3$	$\mu_b(x_2, x_3)$
00	0.25	00	0
01	0.25	01	0.5
10	0.25	10	0
11	0.25	11	0.5

$x_2$	$\mu_2(x_2)$
0	0.5
1	0.5

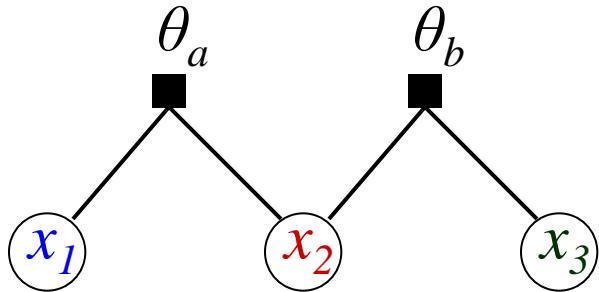
$$\text{MAP: } \min_x \sum_c \theta_c(x_c)$$

$$\text{MAP-LP: } \min_{\mu} \sum_c \sum_{x_c} \mu_c(x_c) \theta_c(x_c)$$

Lower bound on MAP.

# Background: MPLP

---



$x_1 x_2$	$\mu_a(x_1, x_2)$	$x_2 x_3$	$\mu_b(x_2, x_3)$
00	0	00	0
01	1	01	0
10	0	10	0
11	0	11	1

$x_1$	$\mu_1(x_1)$	$x_2$	$\mu_2(x_2)$	$x_3$	$\mu_3(x_3)$
0	1	0	0	0	0
1	0	1	1	1	1

MAP:  $\min_x \sum_c \theta_c(x_c)$   
 MAP-LP:  $\min_{\boldsymbol{\mu}} \sum_c \sum_{\mathbf{x}_c} \mu_c(\mathbf{x}_c) \theta_c(\mathbf{x}_c)$

Lower bound on MAP.  
 $\mathbf{x} \equiv$  integral  $\boldsymbol{\mu}$ .

$$\mathbf{x} = 011$$

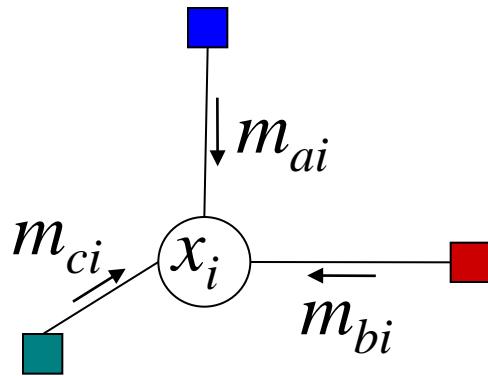
# Background: MPLP

---

Dual LP: sum of beliefs

$$\sum_i \max_{x_i} b_i(x_i)$$

where  $b_i(x_i) = \sum_c m_{ci}(x_i)$



MPLP:

Iteratively update  $m_{ci}(x_i)$

Compute beliefs:  $b_i(x_i) = \sum_c m_{ci}(x_i)$

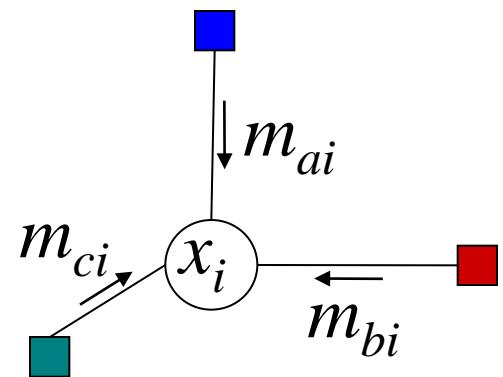
Assign variables:  $x_i^* = \arg \max_{x_i} b_i(x_i)$

# MPLP fixed points

---

Two cases:

- $x^*$  unique - optimal solution
- $b_i(1)=b_i(0)$  for some variables
  - Can find optimal in special cases (e.g. binary  $x$ , pairwise submodular  $\theta$ )
  - Optimal unknown if  $NP$ -hard



$$x_i^* = \arg \max_{x_i} b_i(x_i)$$

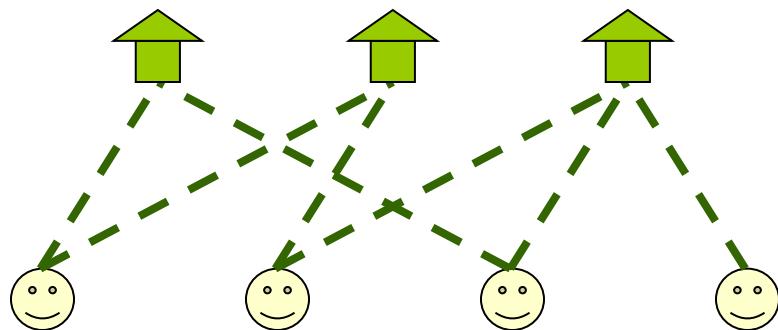
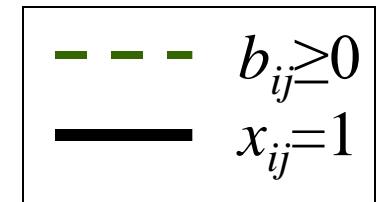
# Complementary slackness

---

- Our approach: **complementary slackness conditions**
  - Always hold for a pair of LP solutions  $(\mu^*, \beta^*)$  that are **primal** and **dual** optimal
- MPLP:  $x^* = \arg \max_x b(x)$ 
  - $\mu^* \equiv x^*$  satisfies a **subset** of c.s. conditions for MAP LP
  - Greedily try satisfy **all** c.s. conditions & achieve the LP lower bound

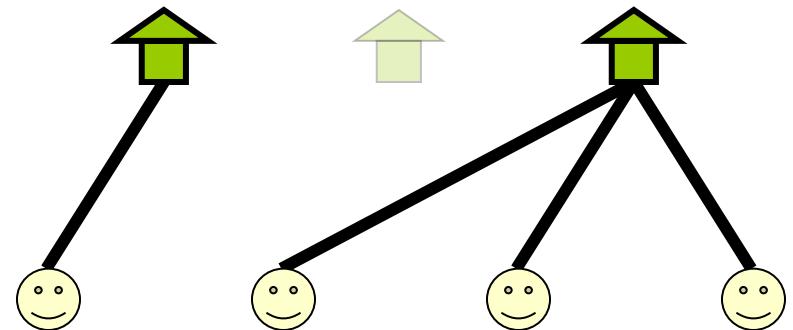
# FL: complementary slackness

Solution support graph  $G=(C,F,E)$ :



MPLP fixed point:  
edges  $b_{ij} \geq 0$

$$b_{ij} \equiv b_{ij}(1) - b_{ij}(0)$$



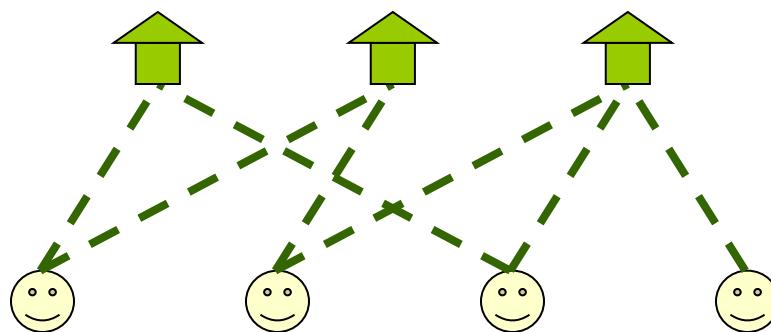
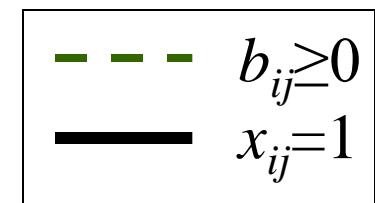
Integral solution  $x$ :  
edges  $x_{ij} = 1$

# FL: complementary slackness

---

Complementary slackness for  $x$ :

1. Customers: connected via an -- edge
2. Facilities: all or no -- edges

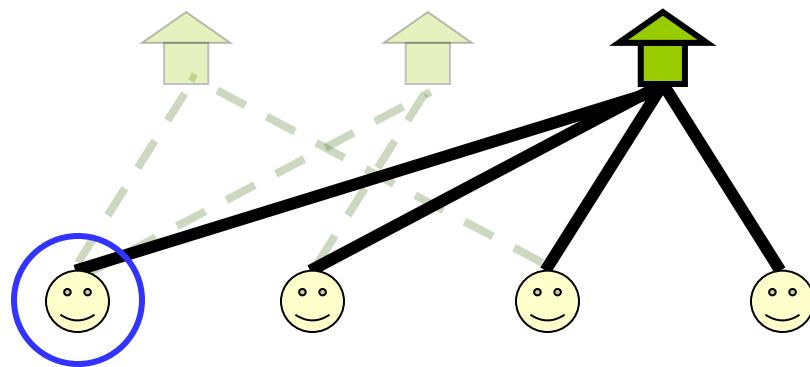


# FL: complementary slackness

Complementary slackness for  $x$ :

1. Customers: connected via an -- edge
2. Facilities: all or no -- edges

---	$b_{ij} \geq 0$
—	$x_{ij} = 1$



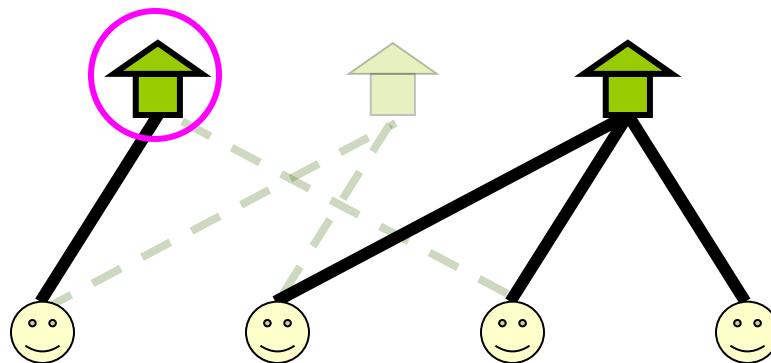
1 violated.

# FL: complementary slackness

Complementary slackness for  $x$ :

1. Customers: connected via an -- edge
2. Facilities: all or no -- edges

---	$b_{ij} \geq 0$
—	$x_{ij} = 1$



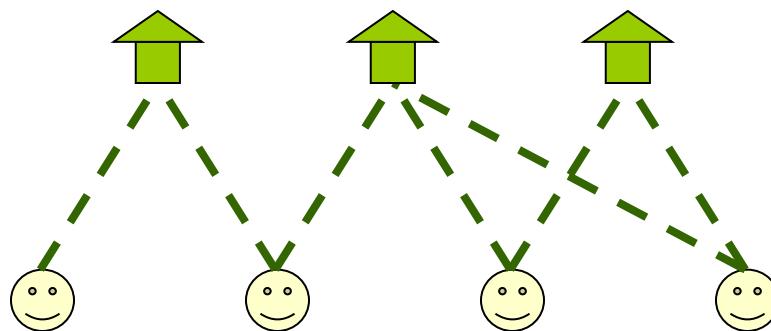
2 violated.

# FL: complementary slackness

Complementary slackness for  $x$ :

1. Customers: connected via an -- edge
2. Facilities: all or no -- edges

---	$b_{ij} \geq 0$
—	$x_{ij} = 1$



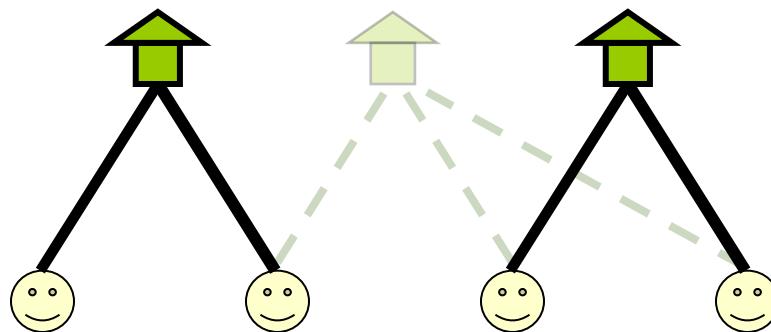
LPr tight – can satisfy both 1 and 2.

# FL: complementary slackness

Complementary slackness for  $x$ :

1. Customers: connected via an -- edge
2. Facilities: all or no -- edges

---	$b_{ij} \geq 0$
—	$x_{ij} = 1$



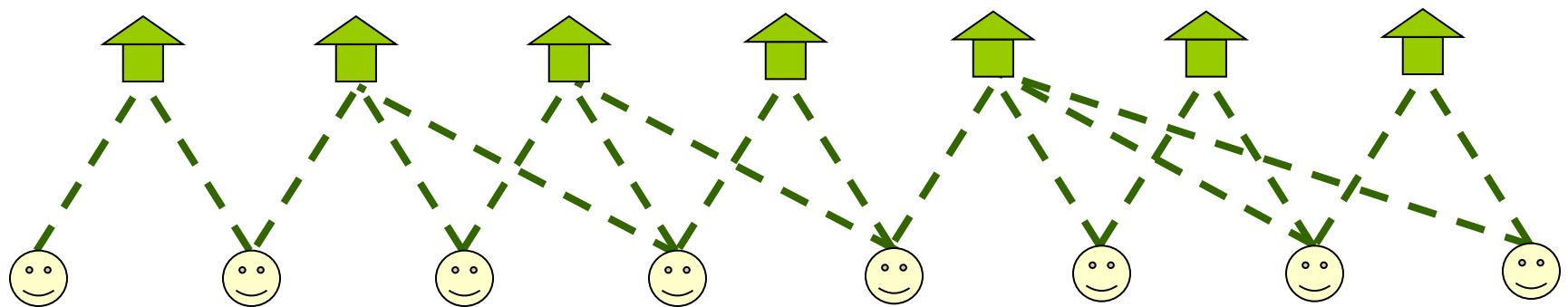
LPr tight – can satisfy both 1 and 2.

# Decoding: belief maximization

---

$$x_{ij}^* = \arg \max_{x_{ij}} b_{ij}(x_{ij})$$

- always satisfies c.s. 1
- pick an edge for each customer

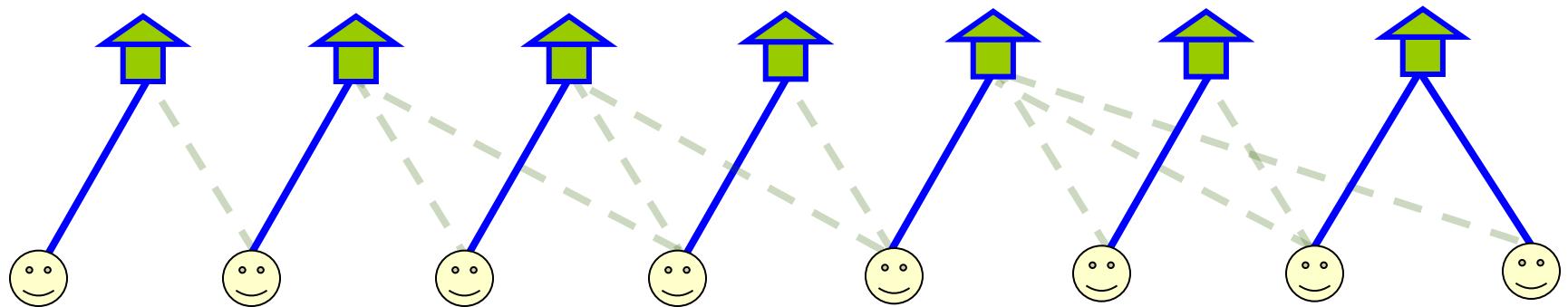


# Decoding: belief maximization

---

$$x_{ij}^* = \arg \max_{x_{ij}} b_{ij}(x_{ij})$$

- always satisfies c.s. 1
- pick an edge for each customer



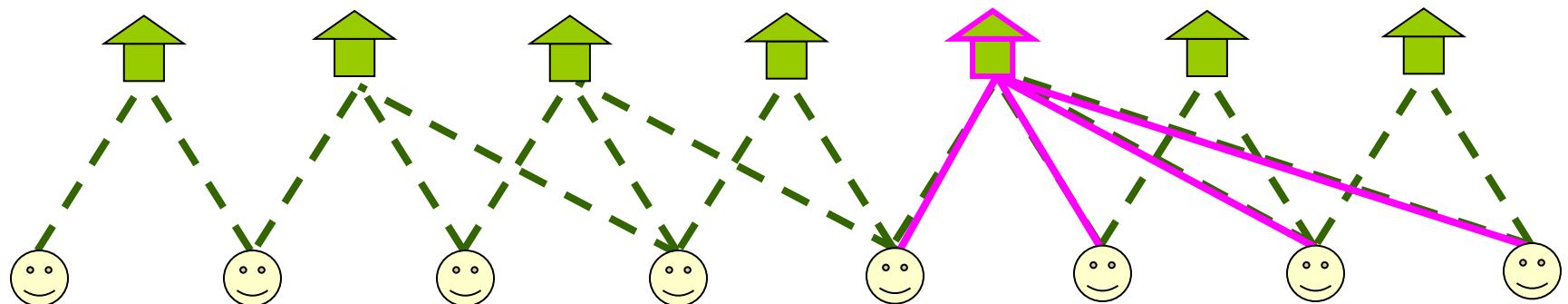
One possible solution – all facilities are open!

# Decoding: greedy algorithm

---

Our approach:

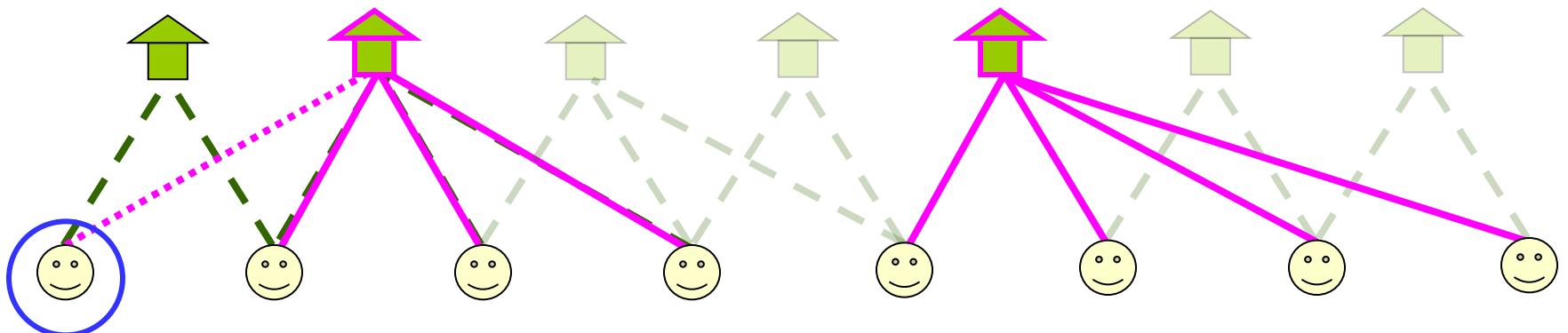
- always satisfy c.s. 2
- open facilities in a greedy order



# Decoding: greedy algorithm

Our approach:

- always satisfy c.s. 2
- open facilities in a greedy order



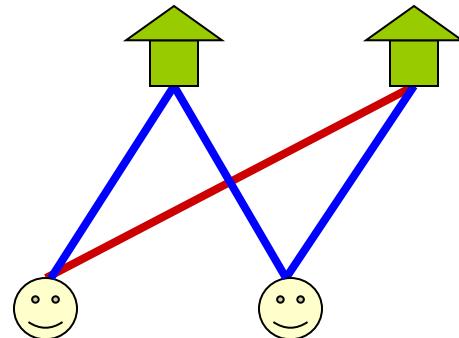
1 violated -  $x_{ij}$  does not maximize its belief

# FL Approximability

---

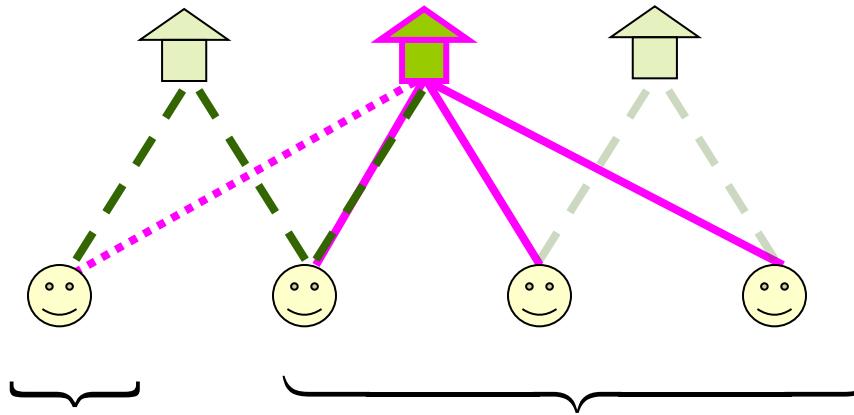
$\rho$ -approximation algorithm: guarantees  $E(\mathbf{x}^*) \leq \rho E(\mathbf{x}^{\text{OPT}})$

- $\rho$  is  $O(\ln/C)$  in general
- $\rho$  is constant for *metric* FL



$$c_{ij} \leq c_{ik} + c_{lk} + c_{lj}$$

# 3-approximation for metric FL

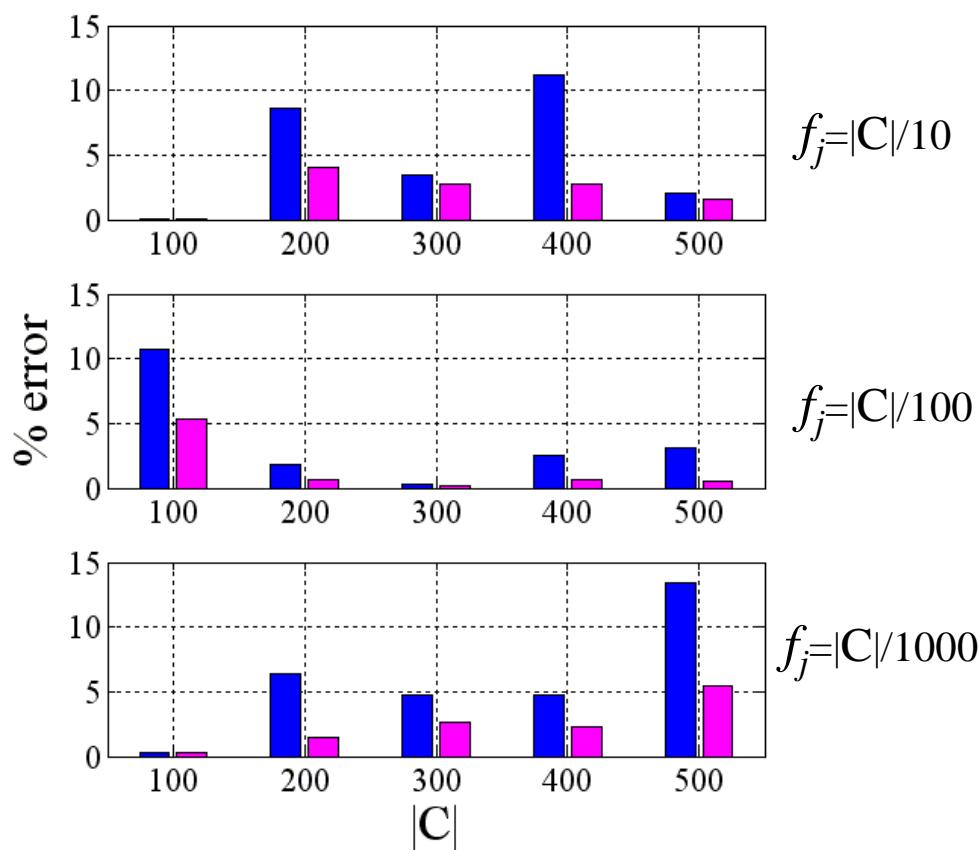


- $\text{Integral} \leq 3 \text{ Dual}$
- Proof: triangle inequality, greedy order, fixed point
- $\text{Integral} = \text{Dual}$
- Proof: complementary slackness

$\text{Integral} \leq 3 \text{ Dual} \leq 3 \text{ Optimal}$

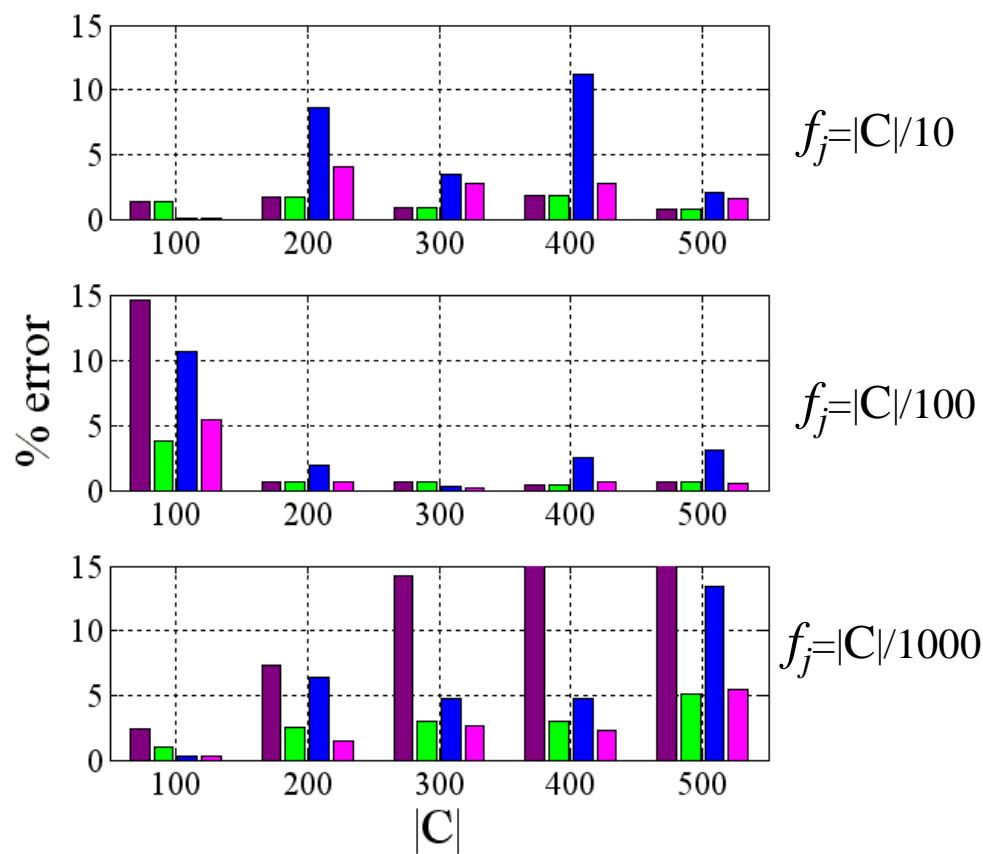
# Experimental evaluation

- Metric FL,  $C=F$ 
  - uniformly sampled 2D points
  - $c_{ij}$ : Euclidean distance
  - $f_j$ : equal for all  $j$
- Algorithms
  - MPLP + Beliefs
  - MPLP + Greedy
- Error: % above LP lower bound



# Experimental evaluation

- Metric FL,  $C=F$
- Standard max-product:  
Affinity Propagation (AP)
- Algorithms
  - AP + Beliefs
  - AP + Greedy
  - MPLP + Beliefs
  - MPLP + Greedy

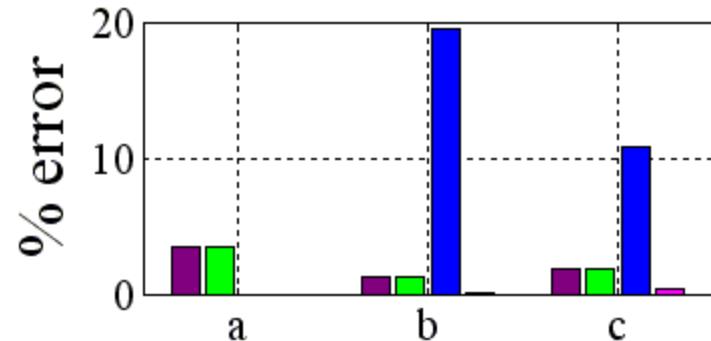


# Experimental evaluation

---

- Non-metric ORLIB benchmarks
- Algorithms
  - MP + Beliefs
  - MP + Greedy
  - MPLP + Beliefs
  - MPLP + Greedy
- Error: % above optimal

Name	C	F	Opt.
c7*	50	16	4/4
c10*	50	25	4/4
c13*	50	50	4/4
a,b,c	1000	100	1/3



# Summary and conclusions

---

- Facility Location
  - Graphical model + MPLP + Greedy decoding
  - 3-approximation for metric FL
  - Improved empirical results over maximizing beliefs
  
- Questions?

