Open problems in throughput scheduling

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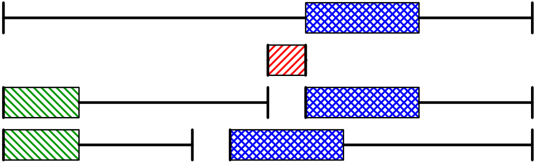
A puzzle



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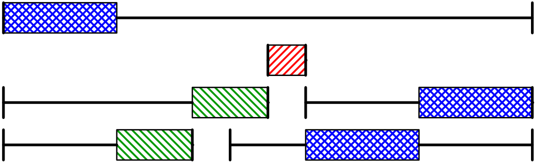

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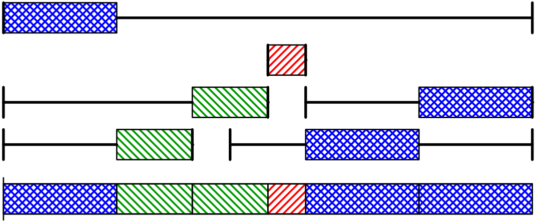

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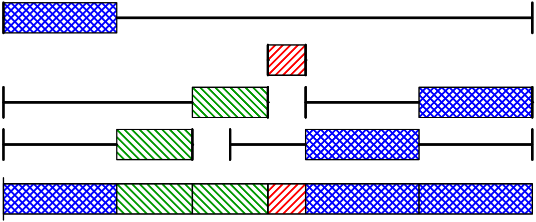

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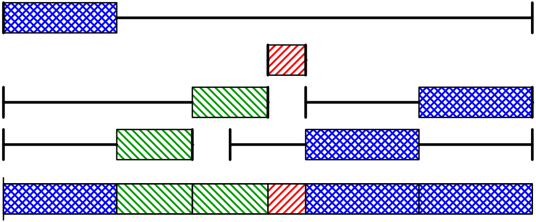

- Move boxes within their ranges.
- Align them so that they do not overlap vertically.
- Is this easy (in P) or difficult (NP-hard)?
- What if there are only two (or 1000) diferent sizes of boxes?

Throughput scheduling

- Environment: One or more machines.
- Input: Jobs with length p_j, release time r_j, deadline d_j, and weight w_j. (Parameters are integers.)
- Output: Each job is assigned to a machine for a subinterval of $[r_j, d_j)$ of length p_j or rejected. No overlaps.
- Objective: Maximize the number (weight) of the completed jobs.

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- Online algorithms.
- Usually a single machine.

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- Online algorithms.
- Usually a single machine.
- Either jobs of equal length ($p_j = p$) and no weights
- or jobs of unit length ($p_j = 1$) with weights.

Online scheduling

- At time r_j, the other parameters of the job become known.
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Competitive ratio

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- $OPT(I) \leq R \cdot A(I)$ for a deterministic algorithm, or
- $OPT(I) \leq R \cdot E[A(I)]$ for a randomized algorithm.

Other scheduling problems

Variants

- Machine environments: speeds, shop scheduling (more operations) etc.
- Job parameters and restrictions: preemption, dependencies, resources etc.

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Typical objectives

- MinMax: Minimize the length of schedule (or another global measure of balance).
- MinSum: Minimize the average completion time of a job (or waiting time, flow time, stretch, possibly weighted).

Jobs of equal length

Setting

- Equal lengths of jobs ($p_j = p$).
- No weights.
- Single machine.

Jobs of equal length

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- No weights.
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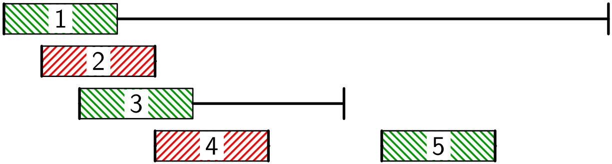
Outline

1. Offline problem is polynomial.
2. Greedy algorithms are 2-competitive.
3. Lower bounds.
4. A better randomized algorithm.
5. Generalizations, variants.

Greedy algorithms

GREEDY: If idle, start an arbitrary job.



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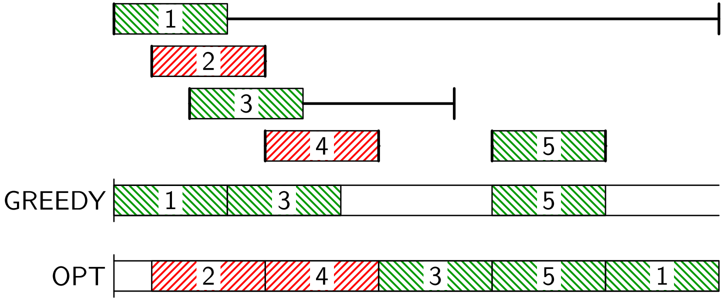

- Any such algorithm is 2-competitive.

Greedy algorithms

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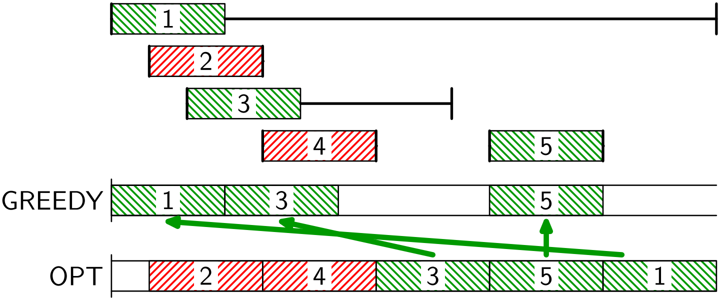

Charging scheme – GREEDY is 2-competitive

- Charge (map) a job in OPT to itself in GREEDY, if scheduled.

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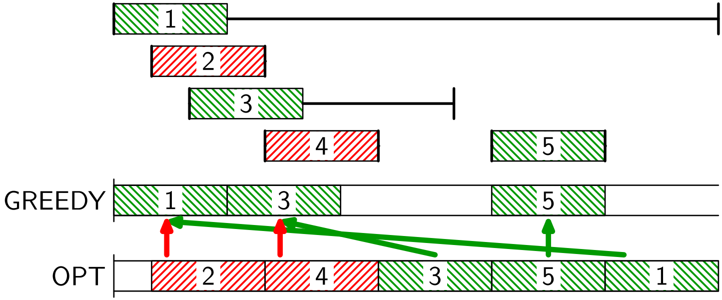
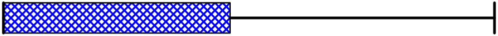


Charging scheme – GREEDY is 2-competitive

- Charge (map) a job in OPT to itself in GREEDY, if scheduled.
- Otherwise charge a job that OPT starts at t to the job GREEDY runs at t.

Lower bounds



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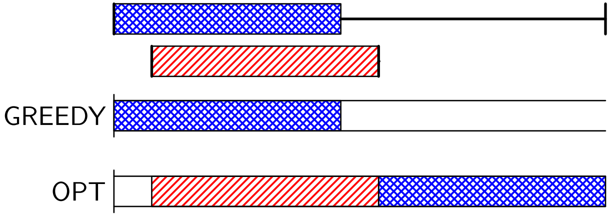

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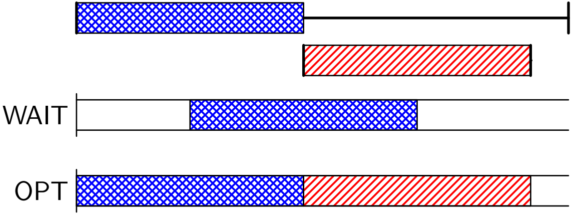

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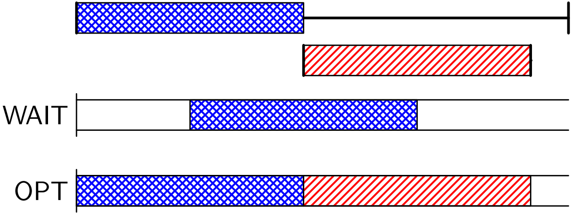
Lower bounds



- No deterministic algorithm is better than 2-competitive.
- No randomized algorithm is better than $4/3$-competitive. (For one of the two instances, on average, runs at most 1.5 jobs out of 2.)

A barely random algorithm I

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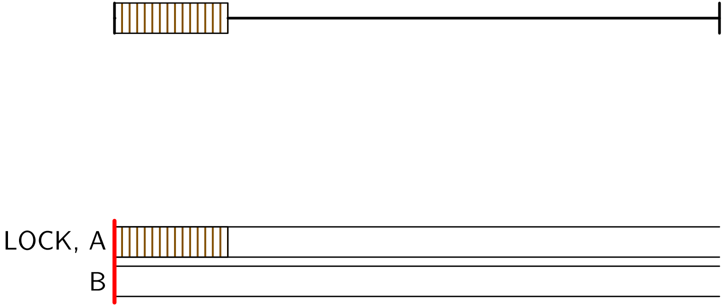
- If the machine is idle (in A or B) and the set of pending jobs is not flexible (idling for time p would lose some job), start the most urgent job.
- If the machine is idle (in A or B) and the set of pending jobs is flexible (idling for time p does no harm):
  - If the lock is available, acquire it, start the most urgent job and release the lock after the job is completed.
  - Otherwise stay idle.

A barely random algorithm II



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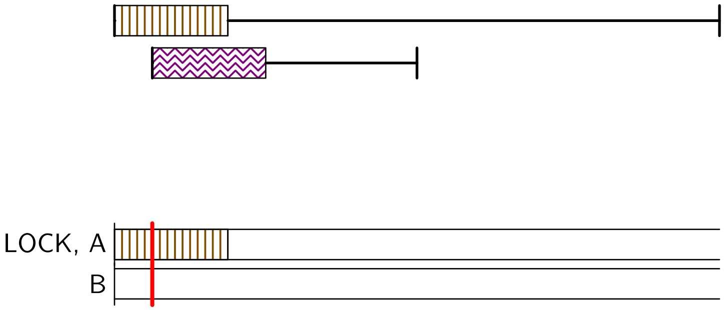

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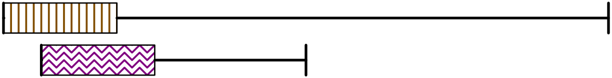

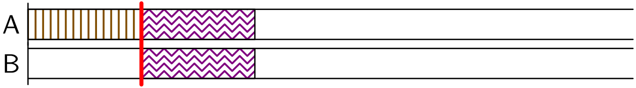

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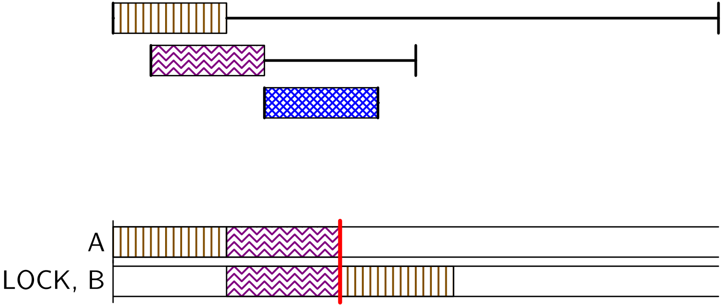

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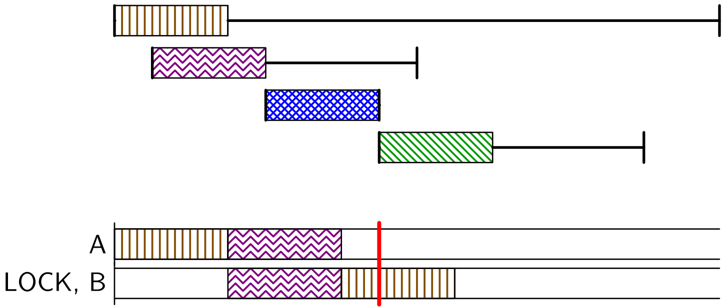

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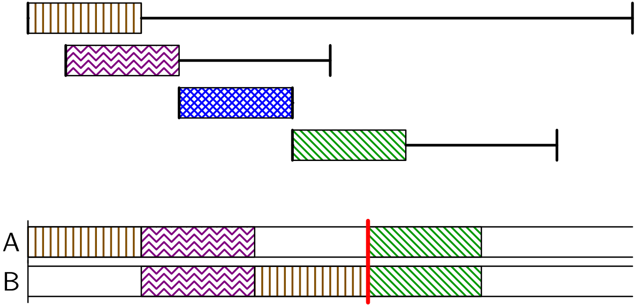

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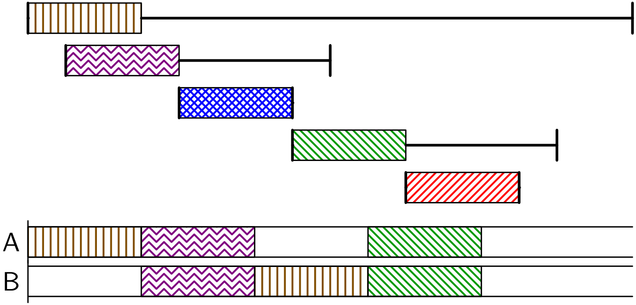

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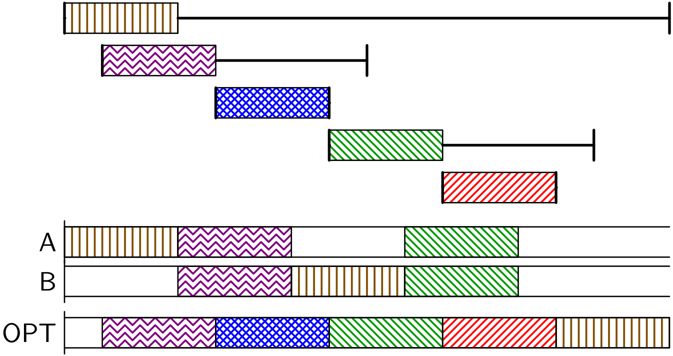

A barely random algorithm III

- Analyzed by a more complex charging scheme.
- Each job in OPT charges $1/2$, $1/3$, or $1/6$ to itself or to the job running at the same time in A and B.
- Each job in A or B is charged at most $5/6$.

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Theorem

This algorithm is $5/3$-competitive.

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Theorem

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Open problem

Find a randomized algorithm with the optimal competitive ratio.

More machines

Parallel machines make the problem easier!

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Results

- For 2 machines, there is a $3/2$-competitive deterministic algorithm and this is optimal.
- For m machines, there is an R-competitive deterministic algorithm with $R \to e/(e-1) \approx 1.58$ for $m \to \infty$.
- The lower bound approaches $6/5$ for $m \to \infty$.

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Open problem

Decrease the gap for $m \to \infty$.

Jobs with fixed start times

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Results

- There is a 4-competitive algorithm for various cases, including equal times ($p_j = p$), unit weights ($w_j = 1$), and uniform weights ($w_j = p_j$); it works for parallel machines.
- There is a matching lower bound.

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GREEDY: Start the released job on the fastest available machine.

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Results for the greedy algorithm

- For two machines, GREEDY is $4/3$-competitive and this is optimal.
- For $m \to \infty$ the competitive ratio is between 1.56 and 2.

Open problem(s)

Analyze GREEDY, or find another algorithm with a competitive ratio below 2.

Unit time jobs with weights

Setting

- Unit length of jobs ($p_j = 1$).
- General weights.
- Single machine.

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Outline

1. Offline problem is easy (matching).
2. Greedy algorithm is 2-competitive.
3. A better randomized algorithm.
4. A better deterministic algorithm.
5. Generalizations, variants.

Motivation and variants

Forwarding packets in network switches

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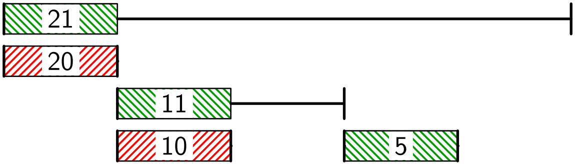
Forwarding packets in network switches

Restricted scenarios

- 2-bounded: Some packets may wait a single step, some packets not at all. ($d_j \leq r_j + 2$)
- Agreeable deadlines: $r_j < r_k$ implies $d_j \leq d_k$.
- Weighted queues: The deadlines are not known, only their order.
- Limited number of weights.

Greedy algorithm

GREEDY: If idle, start a pending job with the maximal weight.



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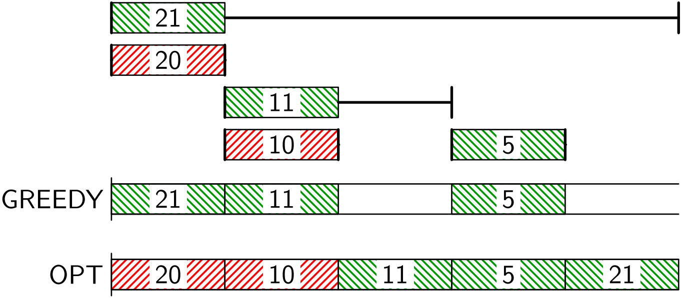

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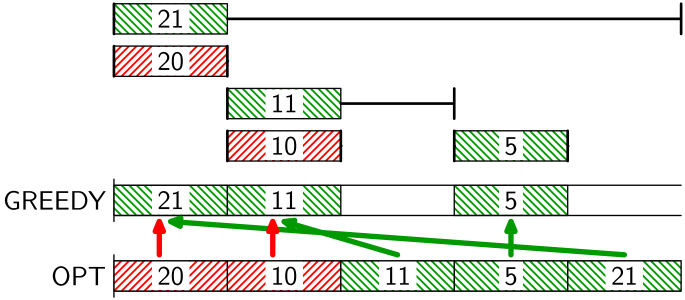

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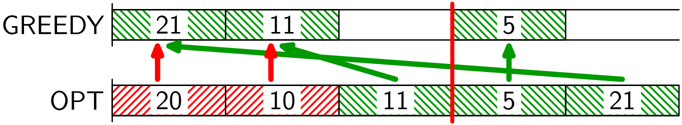
A randomized algorithm

- At each time, pick uniformly random real $x \in (-1, 0)$.
- Let h be the largest weight of a pending job.
- Among all the pending jobs with $w_j \geq e^x \cdot h$, schedule a job with the earlieast deadline.

Theorem

This algorithm is $e/(e-1) \approx 1.58$-competitive.

A potential function



How much "money" we need at a given time and configuration?

We earn $R \cdot w_j$ for running a job and pay w_j if OPT runs a job.

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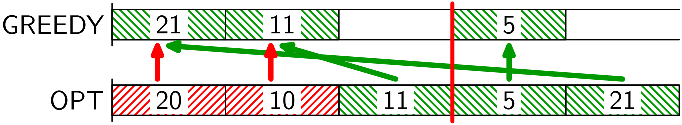

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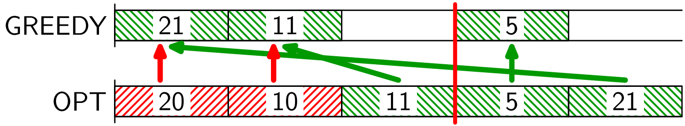

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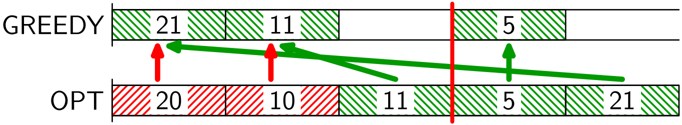

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Analysis

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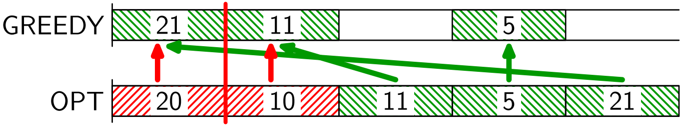

$$\Phi_{old} + R \cdot E[w_{ON}] - w_{OPT} \geq E[\Phi_{new}]$$

Theorem

This algorithm is $e/(e-1) \approx 1.58$-competitive. This is optimal against the adaptive online adversary. I.e., it is optimal among the algorithms analyzed using a potential.

Deterministic algorithms I

Charging scheme

Alternating heavy and urgent packets eventually leads to 1.939-competitive algorithm.

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Charging scheme

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Potential function

Can be used to give a 1.828-competitive algorithm.

Deterministic algorithms II

Modifying the optimal schedule

At each step, the configuration of the optimal schedule is made identical with that of the online algorithm, with some advantage to the optimum:

- Schedule a job and keep it pending,
- Schedule two jobs,
- Increase the weight or deadline of some pending job.

Can be used to give a $\phi \approx 1.618$-competitive algorithm for instances with agreeable deadlines.

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Weighted queues

There exists a 1.897-competitive algorithm.

Lower bounds

2-bounded instances

- The $\phi \approx 1.618$-competitive deterministic algorithm is optimal.
- There exists a 1.25-competitive randomized algorithm and this is optimal.

No other lower bounds for the general problem are known.

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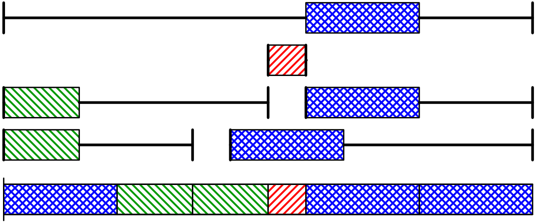
Open problem

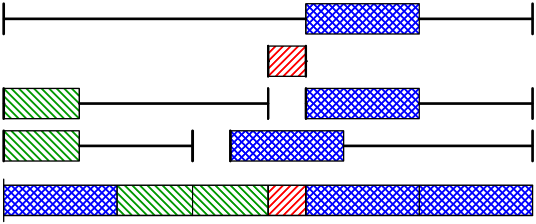
Is the general problem harder than the 2-bounded case?

Offline scheduling



- For unrestricted job lengths, the problem is strongly NP-hard
- For unit jobs ($p_j = 1$) and arbitrary weights we can maximize the weight of scheduled jobs in polynomial time.

Offline scheduling



- Even maximizing the weight for equal-length jobs ($p_j = p$) on a single machine is in P.
- For unit jobs and one more job length ($p_j = \{1, p\}$), we can test if all the jobs can be scheduled.

A linear program

Variables: x_t – the number of long jobs started before time t.
Constraints: For all times s, t:

$$
\begin{aligned}
x_t - x_{t-1} &\geq 0 \\
x_t - x_{t-p} &\leq 1 \\
x_{t+1-p} - x_s &\geq b_{s,t} \\
x_{t+1-p} - x_s &\leq \lfloor (t - s - a_{s,t})/p \rfloor
\end{aligned}
$$

where $a_{s,t}$ and $b_{s,t}$ is the number of short and long jobs, resp., that have to start and complete in $[s, t)$.

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Observation

The matrix of the LP is totally unimodular. Thus if the LP is feasible, then there exists an integral solution.

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- A schedule implies a feasible (integral) solution: Easy.

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$$\begin{aligned} x_t - x_{t-1} &\geq 0 \\ x_t - x_{t-p} &\leq 1 \\ x_{t+1-p} - x_s &\geq b_{s,t} \\ x_{t+1-p} - x_s &\leq \lfloor (t - s - a_{s,t})/p \rfloor \end{aligned}$$

where $a_{s,t}$ and $b_{s,t}$ is the number of short and long jobs, resp., that have to start and complete in $[s, t)$.

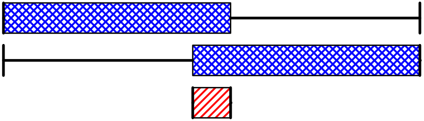
- A schedule implies a feasible (integral) solution: Easy.
- A feasible integral solution implies a schedule: Subtle, holds only for a single machine.

A linear program

Variables: x_t – the number of long jobs started before time t.
Constraints: For all times s, t:

$$\begin{array}{rcl} x_t - x_{t-1} & \geq & 0 \\ x_t - x_{t-p} & \leq & 1 \\ x_{t+1-p} - x_s & \geq & b_{s,t} \\ x_{t+1-p} - x_s & \leq & \lfloor (t - s - a_{s,t})/p \rfloor \end{array}$$

where $a_{s,t}$ and $b_{s,t}$ is the number of short and long jobs, resp., that have to start and complete in $[s, t)$.



Offline scheduling – open problems

Open problems

- If $p_j \in \{2, 3\}$, is it polynomial to decide if all jobs can be scheduled?
- If $p_j \in \{1, 2\}$, is it polynomial to maximize the number of scheduled jobs?

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THANK YOU!