On-Line Learning

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SMART
Statistical Multilingual Analysis
for Retrieval and Translation

Summary

Summary

Background

- Theory of repeated games
  (Hannan, 1956; Blackwell, 1956)
- Compression of individual sequences
  (Lempel and Ziv, 1976)
- Gambling and portfolio selection
  (Cover, 1965 and 1991)
- Pattern classification
  (Novikov, 1962; Littlestone, 1989)

Unifying framework

Prediction with expert advice

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Goal

Bound the number of prediction mistakes without making any statistical assumptions on the way the data sequence is generated

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Competitive analysis

Compare the performance of the forecaster to that of a set of *reference forecasters* (experts)

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Forecaster	1	0	1	1	$M = 2$
Bit sequence	1	1	0	1	

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Goal (refined)

Predict each sequence almost as well as the best expert for that sequence

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$$\ell(\widehat{p}_1, y_1) + \ell(\widehat{p}_2, y_2) + \ldots$$

where $\ell : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a loss function

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- Quadratic loss: $\mathcal{X} = \mathcal{Y} = [0, 1]$ and $\ell(\widehat{p}, y) = (\widehat{p} - y)^2$
- Absolute loss: $\mathcal{X} = [0, 1]$, $\mathcal{Y} = \{0, 1\}$ and $\ell(\widehat{p}, y) = |\widehat{p} - y|$

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Note

Experts are viewed as abstract entities, generating predictions in an unspecified way

Regret

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We want to design consistent forecasters, i.e. such that

$$
\lim_{n \to \infty} \frac{1}{n} \left(\max_{i=1,\dots,N} R_{i,n} \right) = 0
$$

for any sequence of outcomes and all choices of expert advice

Weighted average forecasters

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$$\widehat{p}_t = \frac{\sum_{i=1}^{N} \mu(R_{i,t-1}) f_{i,t}}{\sum_{j=1}^{N} \mu(R_{j,t-1})}$$

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- This is the weighted average forecaster

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Definition

Potential function $\Phi : \mathbb{R}^N \to \mathbb{R}$

$$\Phi(\mathbf{R}) = \psi\left(\sum_{i=1}^{N} \phi(R_i)\right)$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is such that $\psi \geqslant 0, \psi' > 0, \psi'' \leqslant 0$

Blackwell condition

- Using the potential, the prediction at time t gets rewritten as

$$\widehat{p}_t = \frac{\sum_{i=1}^N \nabla\Phi(R_{i,t-1})_i f_{i,t}}{\sum_{j=1}^N \nabla\Phi(R_{j,t-1})_j}$$

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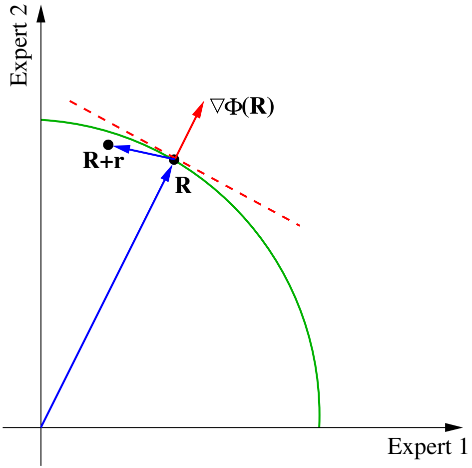
- If the loss is convex, then the following holds

$$\nabla\Phi(\mathbf{R}_{t-1})^\top \mathbf{r}_t \leqslant 0 \qquad \text{(Blackwell condition)}$$

Gradient descent interpretation



Polynomial potential

- Potential function

$$\Phi_p(\mathbf{R}) = \left(\sum_{i=1}^{N} (R_i)_+^p \right)^{2/p} = \|(\mathbf{R})_+\|_p^2 \qquad \text{for } p \geqslant 2$$

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$$\widehat{p}_t = \frac{\sum_{i=1}^N \phi'(R_{i,t-1}) f_{i,t}}{\sum_{j=1}^N \phi'(R_{j,t-1})} = \frac{\sum_{i=1}^N (R_{i,t-1})_+^{p-1} f_{i,t}}{\sum_{j=1}^N \left(R_{j,t-1}\right)_+^{p-1}}$$

Exponential potential

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- Prediction:

$$\widehat{p}_t = \frac{\sum_{i=1}^{N} e^{\eta\left(\widehat{L}_{t-1} - L_{i,t-1}\right)} f_{i,t}}{\sum_{j=1}^{N} e^{\eta\left(\widehat{L}_{t-1} - L_{i,t-1}\right)}} = \frac{\sum_{i=1}^{N} e^{-\eta L_{i,t-1}} f_{i,t}}{\sum_{j=1}^{N} e^{-\eta L_{i,t-1}}}$$

Regret bounds

Loss ℓ is convex and takes values in $[0,1]$

- Polynomial potential with $p = 2 \ln N$

$$\max_{i=1,\dots,N} \frac{R_{i,n}}{n} \leqslant \sqrt{\frac{(2e)}{n} \ln N}$$

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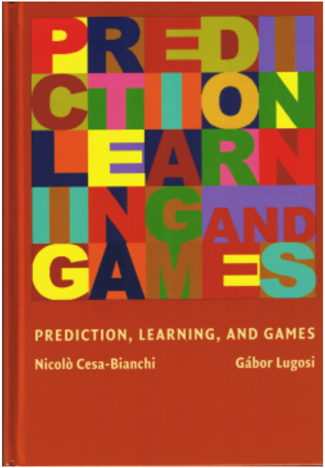
The regret of any forecaster must satisfy:

$$\max_{i=1,\dots,N} \frac{R_{i,n}}{n} = \big(1 - o(1)\big)\sqrt{\frac{2}{n}\ln N}$$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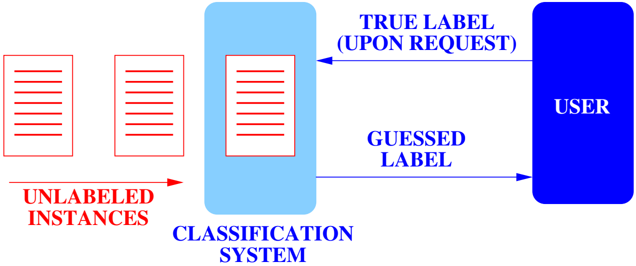
Summary

On-line classification



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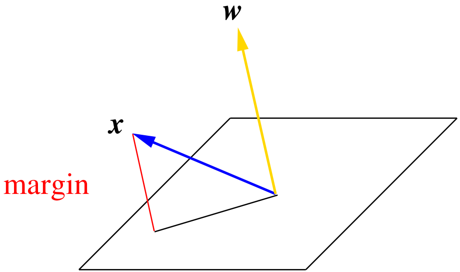

Linear classifiers (cont.)

If $\widehat{p}_t \neq y_t$ then mistake at step t

Goal

On any arbitrary sequence $(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), \dots$ perform not much worse than the best fixed linear classifier

Direct application of experts' framework

One expert for each linear classifier

- Consider the class $\mathcal{F}$ of all linear classifiers $\widehat{p}_t = \textsc{sgn}(\mathbf{u}^\top \mathbf{x}_t)$ for $\mathbf{u} \in \mathbb{R}^d$ with $\|\mathbf{u}\|$ bounded

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- A covering of $\mathcal{F}$ has size exponential in d
- Running the weighted average forecaster on the covering requires managing an exponential number of weights

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- Regret $\quad \boldsymbol{r}_t = y_t \, \boldsymbol{x}_t \, \mathbb{I}_{\{\widehat{p}_t \neq y_t\}}$

A reduction (cont.)

- Weighted average forecaster for binary classification

$$\boldsymbol{w}_{t-1} = \nabla\Phi(\boldsymbol{R}_{t-1}) \qquad \widehat{p}_t = \mathrm{SGN}(\boldsymbol{w}_{t-1}^\top \boldsymbol{x}_t)$$

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- Indeed,

$$w_{t-1}^\top r_t = y_t\, w_{t-1}^\top x_t\, \mathbb{I}_{\{\widehat{p}_t \neq y_t\}} = \begin{cases} 0 & \text{if } \mathbb{I}_{\{\widehat{p}_t \neq y_t\}} = 0 \\ < 0 & \text{otherwise} \end{cases}$$

since $\mathbb{I}_{\{\widehat{p}_t \neq y_t\}} = 1$ iff $\mathrm{SGN}(w_{t-1}^\top x_t) \neq y_t$

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- The potential-based analysis can be adapted to bound the regret against any fixed linear classifier

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If a potential is Legendre, then $\nabla\Phi$ is invertible

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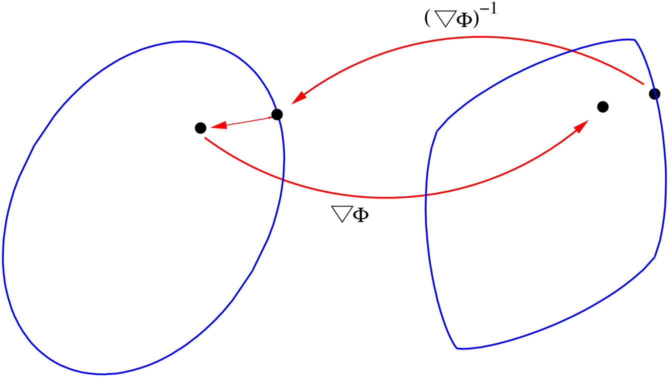
Update rule

$$\boldsymbol{w}_t = \nabla\Phi\Big((\nabla\Phi)^{-1}(\boldsymbol{w}_{t-1}) + y_t\,\boldsymbol{x}_t\,\mathbb{I}_{\{\widehat{p}_t \neq y_t\}}\Big)$$

Incremental formulation (cont.)



$$w_t = \nabla\Phi\Big((\nabla\Phi)^{-1}(w_{t-1}) + y_t\, x_t\, \mathbb{I}_{\{\widehat{p}_t \neq y_t\}} \Big)$$

Application to polynomial potential

- Polynomial potential $\Phi_p(\cdot) = \|\cdot\|_p^2$ is Legendre

$$\left(\nabla \tfrac{1}{2}\|\mathbf{u}\|_p^2\right)_i = \frac{\mathrm{SGN}(u_i)\,|u_i|^{p-1}}{\|\mathbf{u}\|_p^{p-2}} \qquad \left(\nabla \tfrac{1}{2}\|\mathbf{u}\|_p^2\right)^{-1} = \nabla \tfrac{1}{2}\|\mathbf{u}\|_q^2$$

where q is such that $1/p + 1/q = 1$

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- When $p = 2$ we have $\nabla\Phi_2(\mathbf{R}) = \mathbf{R}$

Application to polynomial potential

- Polynomial potential $\Phi_p(\cdot) = \|\cdot\|_p^2$ is Legendre

$$\left(\nabla \tfrac{1}{2}\|\mathbf{u}\|_p^2\right)_i = \frac{\mathrm{SGN}(u_i)\,|u_i|^{p-1}}{\|\mathbf{u}\|_p^{p-2}} \qquad \left(\nabla \tfrac{1}{2}\|\mathbf{u}\|_p^2\right)^{-1} = \nabla \tfrac{1}{2}\|\mathbf{u}\|_q^2$$

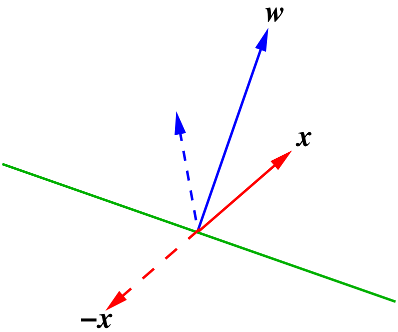
where q is such that $1/p + 1/q = 1$

- When $p = 2$ we have $\nabla\Phi_2(\mathbf{R}) = \mathbf{R}$
- The update rule then is simply

$$\boldsymbol{w}_t = \boldsymbol{w}_{t-1} + y_t\,\boldsymbol{x}_t\,\mathbb{I}_{\{\widehat{p}_t \neq y_t\}}$$

the Perceptron algorithm (Rosenblatt, 1952)

The Perceptron algorithm



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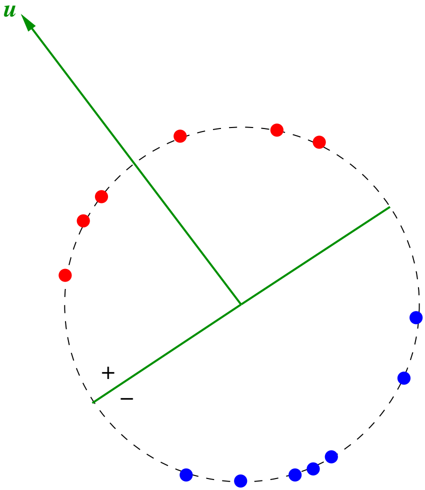
- This is the Winnow algorithm (Littlestone, 1988)

The linearly separable case



Comparison between poly. and exp. potential

Mistake bounds for linearly separable sequences

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Mistake bounds for linearly separable sequences

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- For $p \approx 2\ln d$ the bounds are essentially equal

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- Consider a sequence $(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2) \dots$ such that $\boldsymbol{x}_t \in \{-1, 1\}^d$ and $y_t = \text{SGN}(x_{1,t})$

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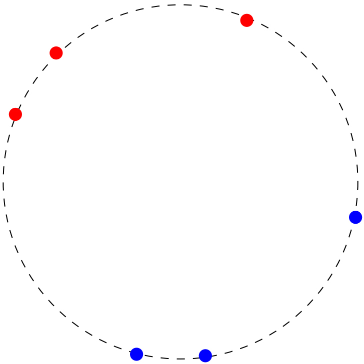
- Opposite situation when instances $\mathbf{x}_t$ are sparse and best expert $\mathbf{u}$ is dense

The cone of consistent hyperplanes



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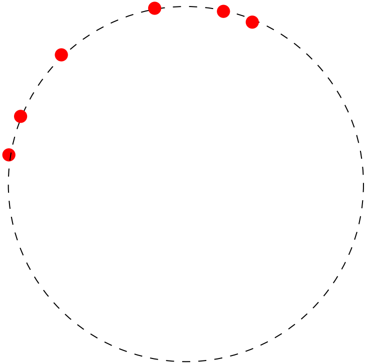

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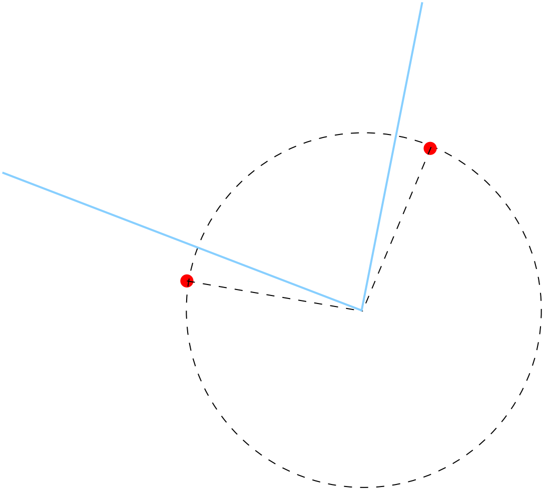

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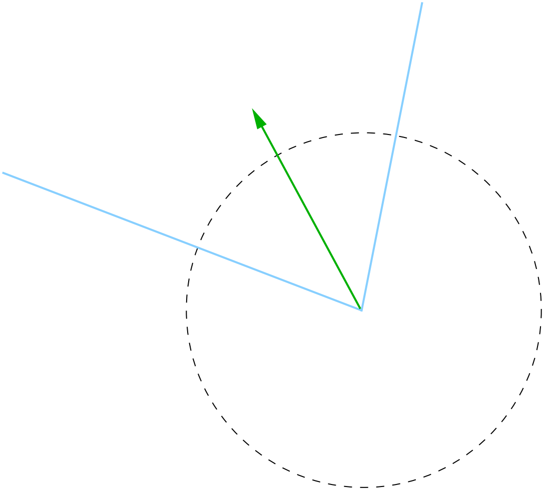

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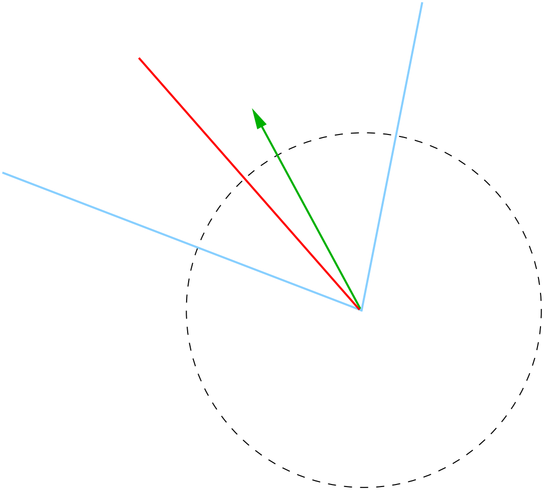

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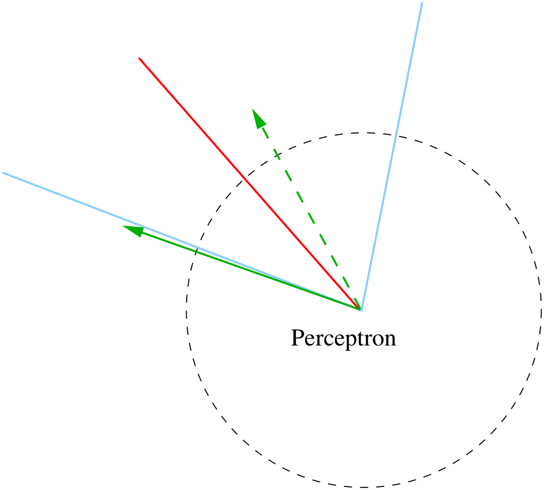

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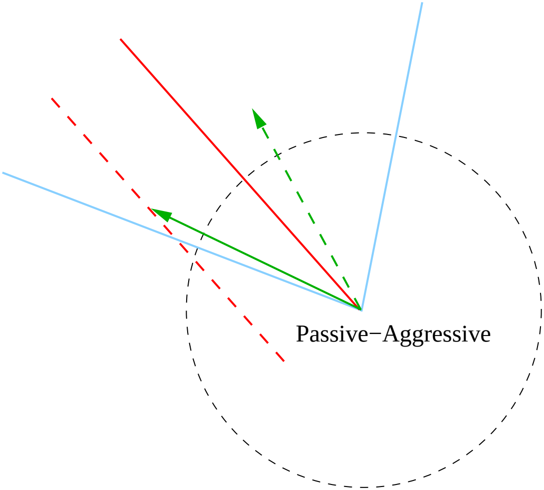

Mistake bounds for various updates

On any sequence of examples such that $y_t \mathbf{u}^\top \mathbf{x}_t \geqslant 1$ with $\|\mathbf{u}\| = U$

bound	algorithm	update time
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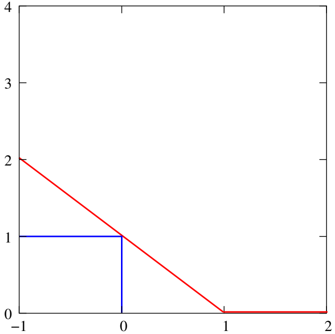
$d \ln U$	Geometric center	$O(d^4)$

The nonseparable case



$$\underbrace{\mathbb{I}_{\{\mathrm{SGN}(z)\neq y\}}}_{\text{mistake ind.}} \leqslant \underbrace{\left(1 - y\,z\right)_+}_{\text{hinge loss}}$$

- Computing an hyperplane minimizing the number of misclassified examples is NP-hard
- The hinge loss is a convex upper bound of the mistake indicator function

Perceptron mistake bound

Perceptron's performance is compared to the hinge loss of the single best linear classifier $\mathbf{u} \in \mathbb{R}^d$ in hindsight

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Similar to the SVM functional $$\inf_{\mathbf{u} \in \mathbb{R}^d} \left(D_{\mathbf{u}} + \|\mathbf{u}\|^2 \right)$$

Summary

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- Then we can learn $w = \sum_i \alpha_i \phi(\boldsymbol{x}_{t_i})$ in the RKHS because

$$\begin{aligned}
\textsc{sgn}\big(\langle w, \phi(\boldsymbol{x})\rangle\big) &= \textsc{sgn}\left(\sum_i y_{t_i} \left\langle \phi(\boldsymbol{x}_{t_i}), \phi(\boldsymbol{x}) \right\rangle\right) \\
&= \textsc{sgn}\left(\sum_i y_{t_i} K(\boldsymbol{x}_{t_i}, \boldsymbol{x})\right)
\end{aligned}$$

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Perceptron's potential is spherical $\longrightarrow$ rotational invariance

Kernel Perceptron

Start with empty cache $\mathcal{L}$ of examples

Loop:

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Mistake bounds hold in the whole RKHS

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- Thus $B \geqslant U^2$ is necessary to compete against $\mathbf{u}$ of length U
- Can we compete against any $\mathbf{u}$ with $\|\mathbf{u}\| \leqslant U$ using $B = (1+\varepsilon)U^2$ supports?

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Randomized Budget Perceptron

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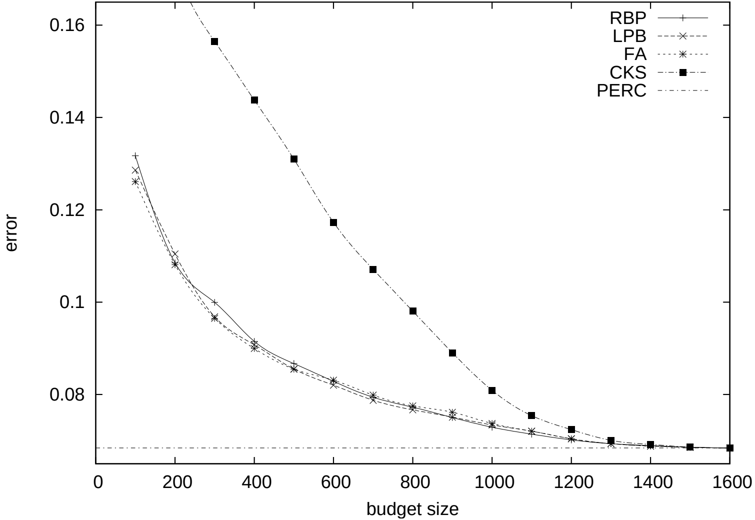
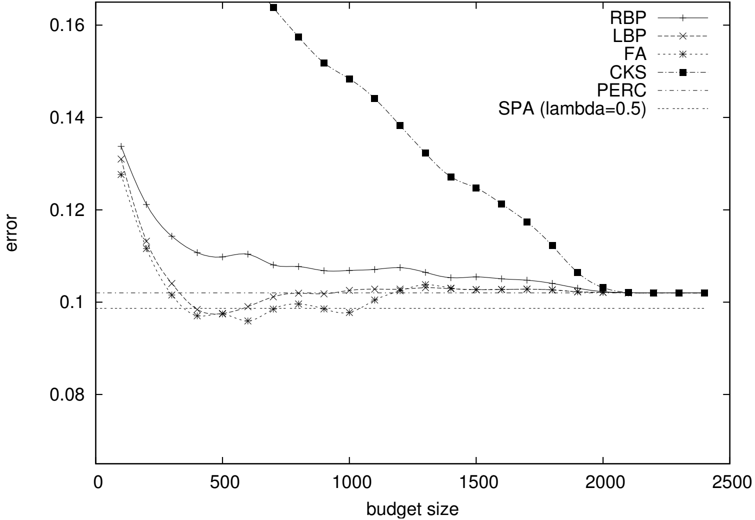
Result:

Bound on mistakes scales roughly with $1 + 1/\varepsilon$

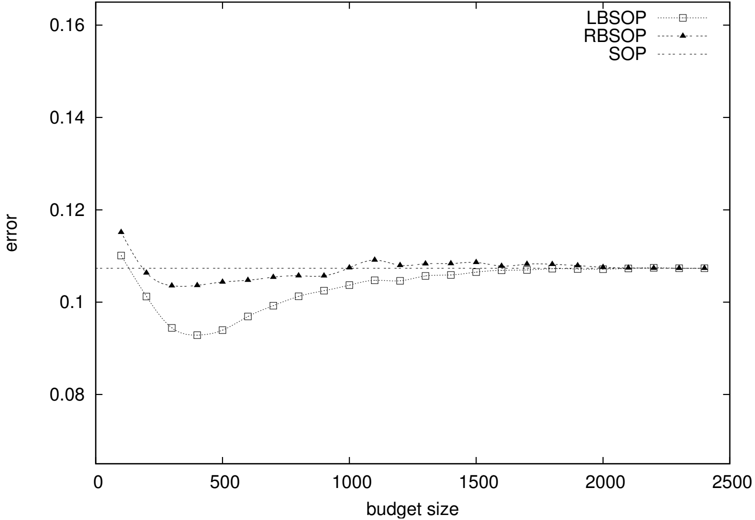
Empirical performance — stationary



Empirical performance — nonstationary



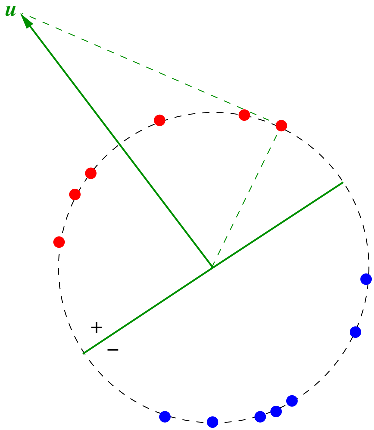
Empirical performance 2nd order — nonstationary



Summary

Online approximation of SVM hyperplane

The SVM hyperplane is the shortest $\mathbf{u}$ such that $y_t \mathbf{u}^\top \mathbf{x}_t \geqslant 1$ for all t



Online approximation of SVM hyperplane (cont.)

The ALMA algorithm

Parameter: $0 < \alpha \leqslant 1$

Set mistake counter $k = 1$

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Loop:

1. Read next instance $\boldsymbol{x}_t$

Online approximation of SVM hyperplane (cont.)

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Parameter: $0 < \alpha \leqslant 1$

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Loop:

1. Read next instance $\boldsymbol{x}_t$
2. Predict y_t with $\widehat{p}_t = \textsc{sgn}\big(\boldsymbol{w}^\top \boldsymbol{x}_t\big)$

Online approximation of SVM hyperplane (cont.)

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Loop:

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3. Obtain true label y_t

Online approximation of SVM hyperplane (cont.)

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4. If margin smaller than $c(1-\alpha)/\sqrt{k}$ then:

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Online approximation of SVM hyperplane (cont.)

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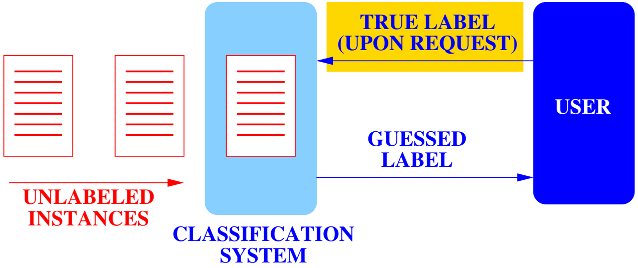
Result

Finds separating $\boldsymbol{u}$ with $\|\boldsymbol{u}\| \leqslant \|\boldsymbol{u}_{\text{SVM}}\| \big/ (1-\alpha)$ after at most $\big(\|\boldsymbol{u}_{\text{SVM}}\| \big/ \alpha\big)^2$ updates

Selective sampling



A selective sampling classifier

1. Classify next instance $\boldsymbol{x}_t$ with $\operatorname{SGN}(\boldsymbol{w}^\top \boldsymbol{x}_t)$

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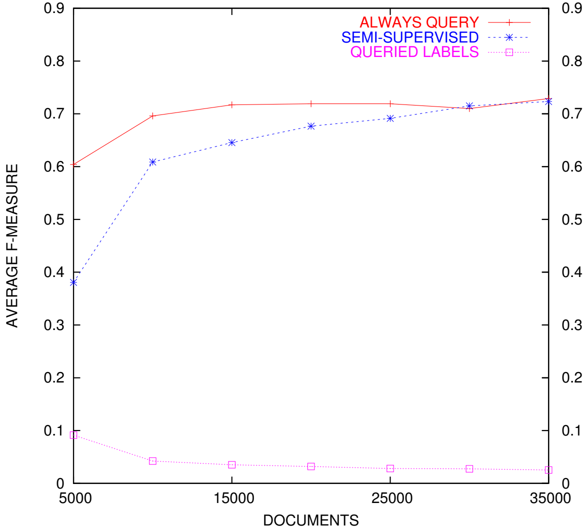
3. If label queried then use $(\boldsymbol{x}_t, y_t)$ to update $\boldsymbol{w}$

N_t = number of labels sampled so far

$\boldsymbol{w}$ updated with the 2nd order Perceptron update rule

Empirical performance on RCV1



Summary

1. Prediction with expert advice
2. Linear classification
3. Kernel-based on-line learning
4. Online SVM and active learning
5. From mistake to risk bounds

Statistical learning theory

- Linear classifiers $H(\boldsymbol{x}) = \mathrm{SGN}\big(\boldsymbol{w}^\top \boldsymbol{x}\big)$

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- $\mathtt{risk}(H) = \mathbb{P}\big(H(\boldsymbol{x}) \neq y\big)$
- Learning algorithm

$$(\boldsymbol{x}_1, y_1), \dots, (\boldsymbol{x}_n, y_n) \longrightarrow \boxed{A} \longrightarrow \widehat{H} : \mathbb{R}^d \to \{-1, +1\}$$

$\widehat{H}$ is (random) hypothesis output by learner

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Goals

1. Bound the average risk of the ensemble in terms of the size of the ensemble
2. Find an element of the ensemble whose risk is close to the ensemble average

Step 1: bound the average risk

The difference

$$\mathtt{risk}(H_{t-1}) - \mathbb{I}_{\{H_{t-1}(x_t) \neq y_t\}}$$

is a martingale difference sequence because

$$\mathbb{E}\Big[\mathtt{risk}(H_{t-1}) - \mathbb{I}_{\{H_{t-1}(x_t) \neq y_t\}} \,\Big|\, (x_1, y_1), \dots, (x_{t-1}, y_{t-1})\Big] = 0$$

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The associated martingale is

$$\sum_{t=1}^{n} \left(\mathrm{risk}(H_{t-1}) - \mathbb{I}_{\{H_{t-1}(x_t) \neq y_t\}}\right)$$

$$\Longleftrightarrow \quad \underbrace{\frac{1}{n} \sum_{t=1}^{n} \mathrm{risk}(H_{t-1})}_{\text{average risk}} - \underbrace{\frac{1}{n} \sum_{t=1}^{n} \mathbb{I}_{\{H_{t-1}(x_t) \neq y_t\}}}_{\text{fraction of mistakes}}$$

Bernstein's bound

If $Z_1, Z_2, \dots$ is a martingale difference sequence with increments bounded by 1 and

$$V_n = \sum_{t=1}^{n} \mathbb{E}\left[Z_t^2 \mid Z_1, \dots, Z_{t-1}\right]$$

then for all $S, K > 0$

$$\mathbb{P}\left(\sum_{t=1}^{n} Z_n \geqslant S, \quad V_n \leqslant K\right) \leqslant \exp\left(-\frac{S^2}{2(S/3 + K)}\right)$$

Application of Bernstein's bound

Since $0 \leqslant \mathbb{I}_{\{H(x) \neq y\}} \leqslant 1$,

$$
\begin{aligned}
& \mathrm{VAR}\left[\mathbb{I}_{\{H_{t-1}(x_t),y_t\}} \,\middle|\, (x_1,y_1),\dots,(x_{t-1},y_{t-1})\right] \\
\leqslant \quad & \mathbb{E}\left[\mathtt{risk}(H_{t-1}) \,\middle|\, (x_1,y_1),\dots,(x_{t-1},y_{t-1})\right]
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Applying Bernstein's gives

$$\frac{1}{n}\sum_{t=1}^{n} \texttt{risk}(H_{t-1}) \quad \leqslant \quad \frac{M_n}{n} + \frac{c}{n}\left(\ln M_n + \sqrt{M_n \ln M_n}\right) \quad \text{w.h.p.}$$

Where $\displaystyle \frac{M_n}{n} = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}_{\{H_{t-1}(x_t) \neq Y_t\}}$ is the fraction of mistakes

Step 2: pick a good classifier in the ensemble

- Start from the ensemble $H_0, H_1, \dots, H_n$
- Do the following:
    1. test each H_t on $(\boldsymbol{x}_{t+1}, y_{t+1}), \dots, (\boldsymbol{x}_n, y_n)$
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Guaranteed bound

$$\mathtt{risk}(\widehat{H}) \quad \leqslant \quad \frac{M_n}{n} + \frac{c}{n}\left((\ln n)^2 + \sqrt{M_n \ln n}\right) \qquad \text{w.h.p.}$$

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