A QCQP Approach to Triangulation

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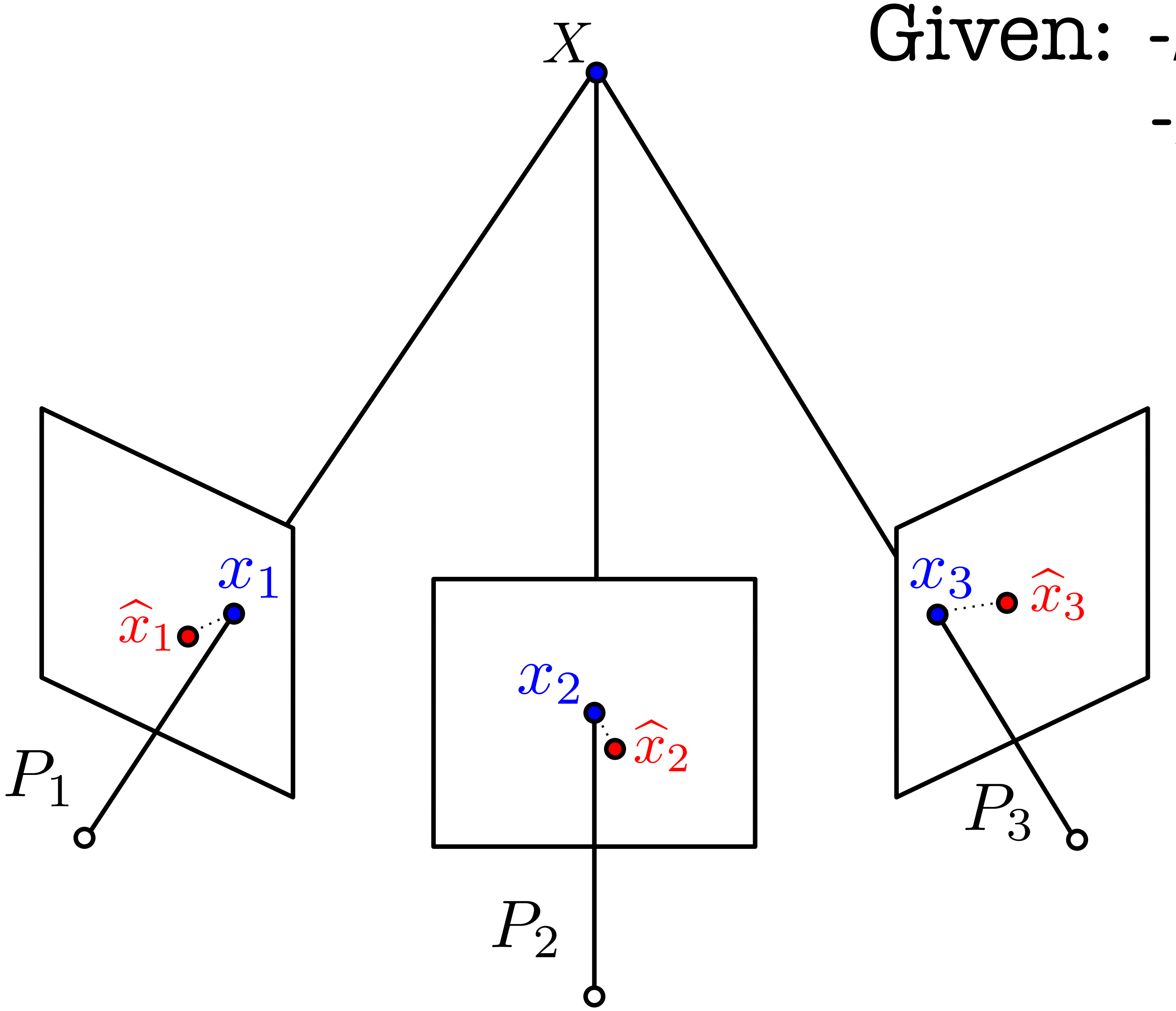
The Triangulation Problem



Given: -n camera matrices $P_i \in \mathbb{R}^{3\times 4}$

-n noisy observations $\widehat{x}_i \in \mathbb{R}^2$

$$P_i = \begin{bmatrix} a_i^\top \\ b_i^\top \\ c_i^\top \end{bmatrix} \qquad \widetilde{X} = \begin{bmatrix} X \\ 1 \end{bmatrix}$$

$$\min_{X \in \mathbb{R}^3} \sum_{i=1}^{n} \|x_i - \widehat{x}_i\|^2$$

$$\text{s.t.} \quad x_i = \begin{bmatrix} \dfrac{a_i^\top \widetilde{X}}{c_i^\top \widetilde{X}} & \dfrac{b_i^\top \widetilde{X}}{c_i^\top \widetilde{X}} \end{bmatrix}^\top$$

Previous Work

First order optimality

- Linear initialization + non-linear refinement
- [Hartley, Seo 08] - Verify global optimality of local solution
- [Hartley, Sturm 97] - Two-view triangulation
- [Stewenius, et al. 05] - Groebner basis of local solution space

Relaxations

- [Kahl, et al. 08] - Branch and bound
- [Kahl, Henrion 07] - SDP relaxations

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Not extendable to higher n

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Exponential in precision or in n

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Our Method

- SDP relaxation
- Pay attention to algebraic structure
- Polynomial in n

Unconstrained to Constrained

$$\min_{X \in \mathbb{R}^3} \sum_{i=1}^{n} \|x_i - \widehat{x}_i\|^2$$

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$\updownarrow$

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$$\text{s.t.} \quad x = (x_1, \ldots, x_n) \in V$$

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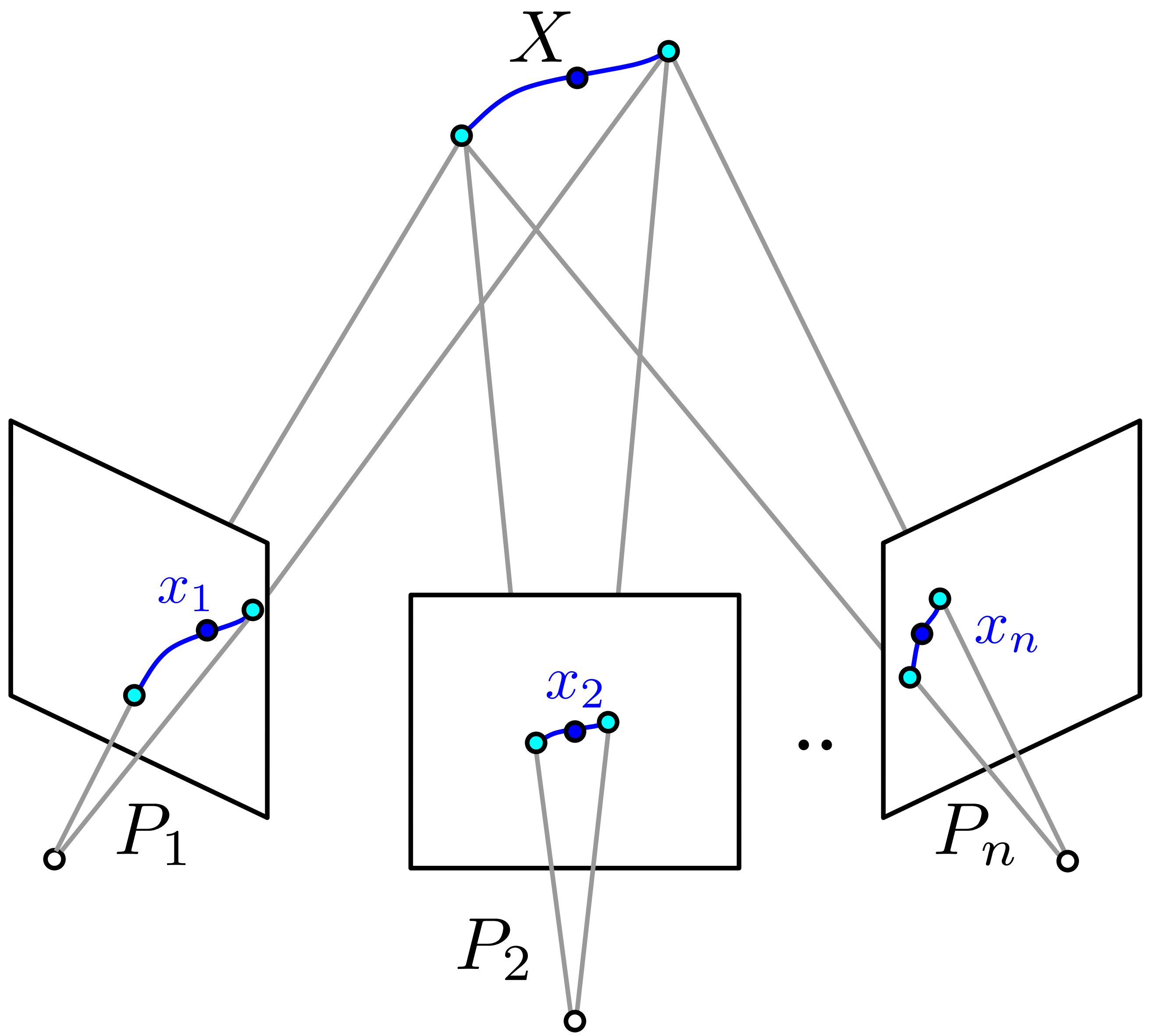

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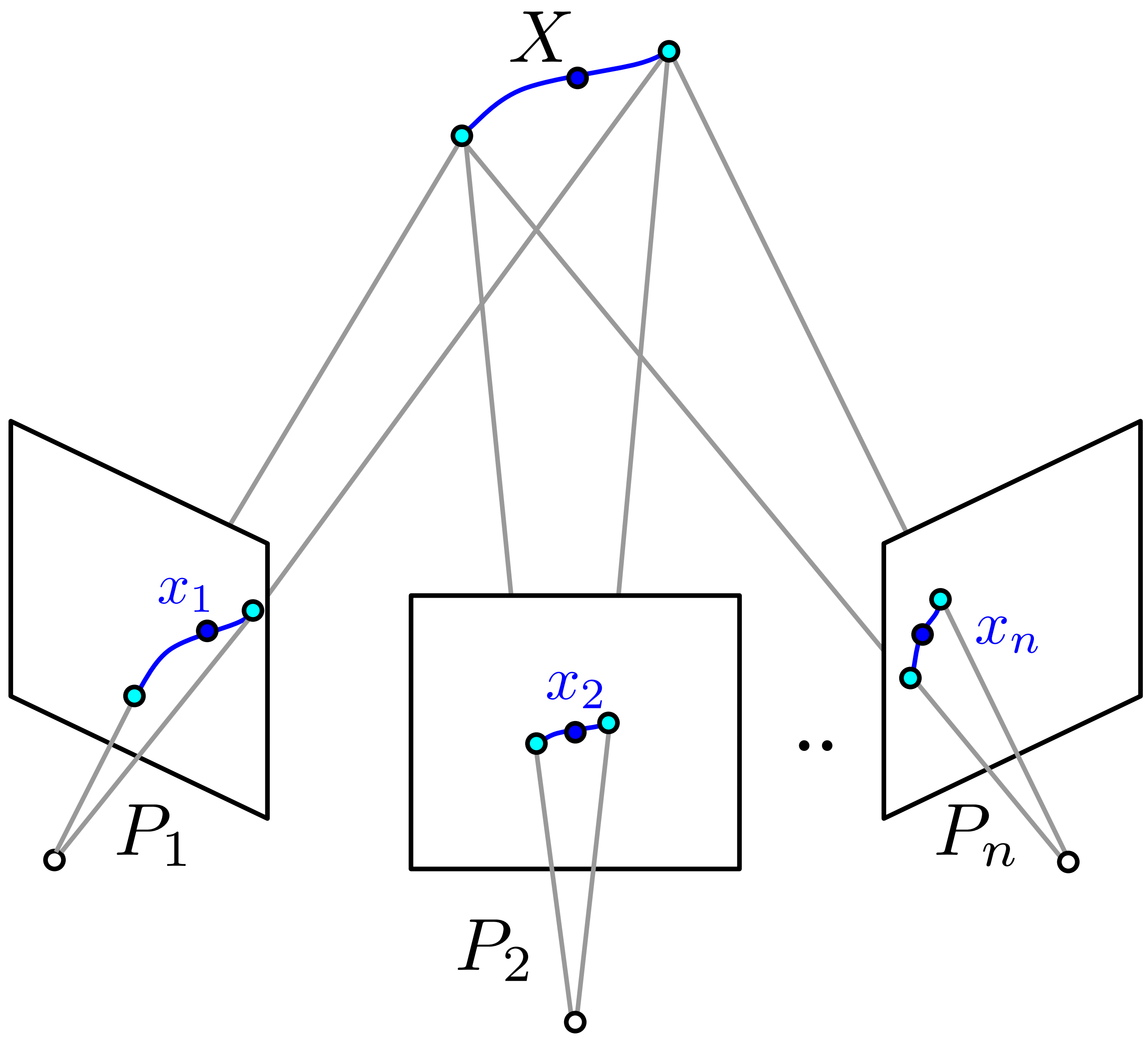

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Can we talk about V without reference to X?

The Constraint Set

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<u>$n = 2$</u>

$$V = \{ (x_1, x_2) \ : \ \widetilde{x}_1^\top F_{12} \widetilde{x}_2 = 0 \}$$

epipolar equation (quadratic)

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quadratic — cubic

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quadratic cubic

$$W = \{(x_1, \ldots, x_n) \ : \ \widetilde{x}_i^\top F_{ij}\, \widetilde{x}_j = 0\}$$

Relaxation of V

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quadratic   cubic

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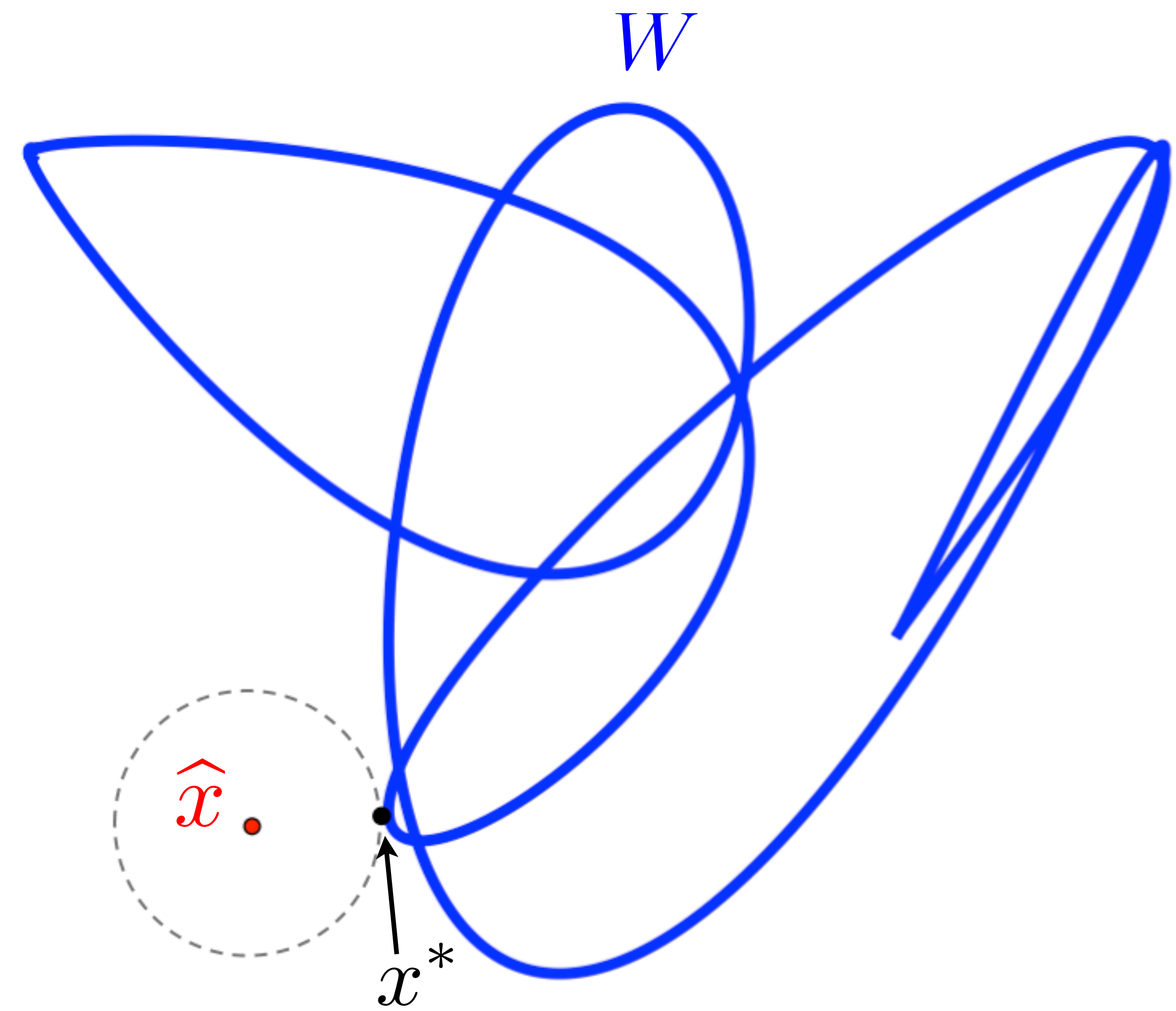
Relaxation of V

[Heyden, Åström 97] $V = W$ when $n = 2$ or non-coplanar cameras.

From QCQP to SDP

$$\min_{x} \quad \|x - \widehat{x}\|^2 \quad \text{s.t.} \quad x \in W \quad \rightarrow x^*$$



Finding the closest point from $\widehat{x}$ to W.

From QCQP to SDP

$$\min_{x} \quad \|x - \widehat{x}\|^2$$
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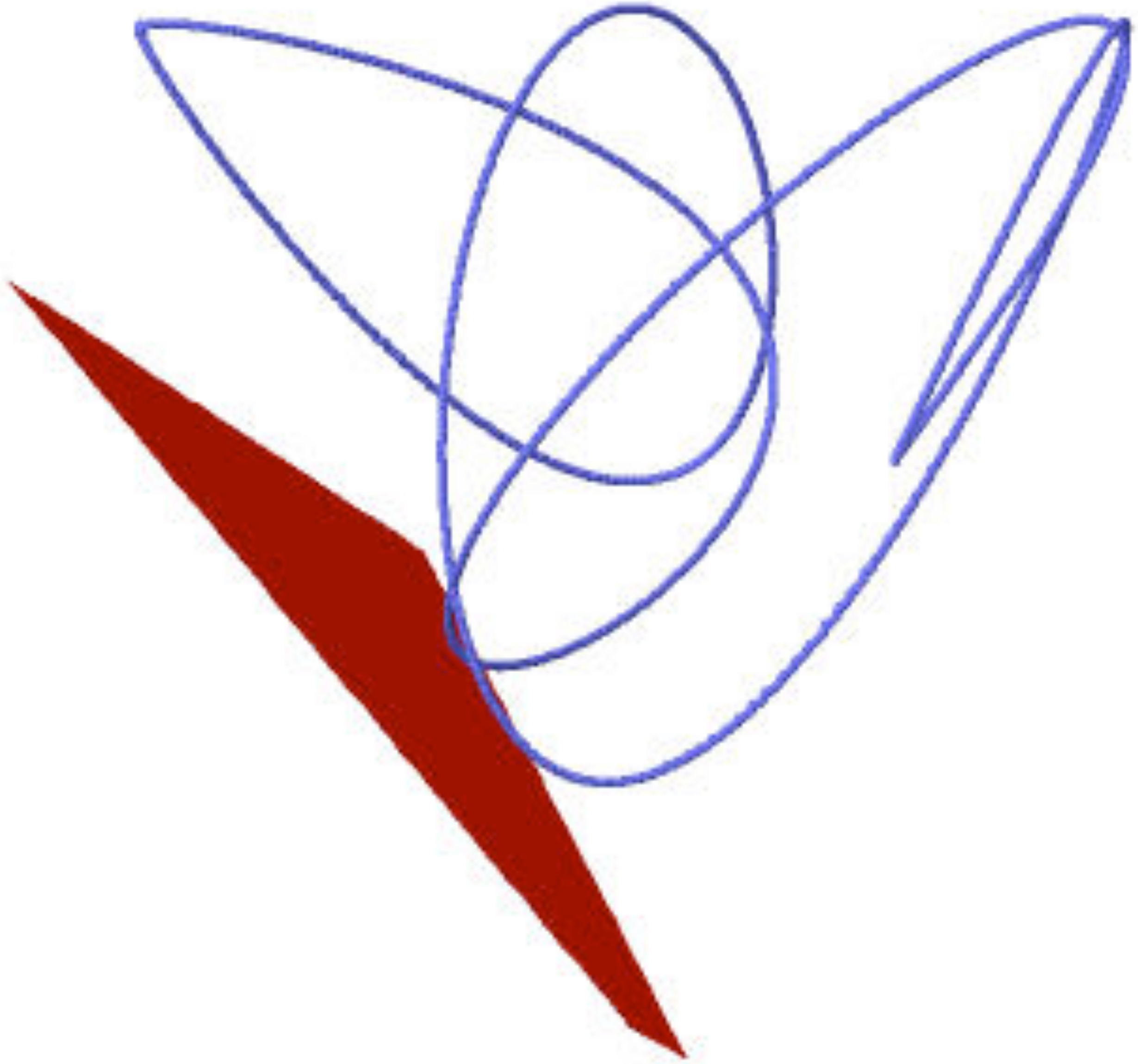
$\rightarrow x^*$

$\updownarrow$

$$\min_{Y} \quad \langle G, Y \rangle$$
$$\text{s.t.} \quad \langle F_i, Y \rangle = b_i$$
$$Y \succeq 0$$
$$\text{rank}(Y) = 1$$



Rank-constrained semidefinite program.

From QCQP to SDP

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$$\begin{aligned} \min_{Y} \; & \langle G, Y \rangle \\ \text{s.t.} \; & \langle F_i, Y \rangle = b_i \\ & Y \succeq 0 \\ & \operatorname{rank}(Y) = 1 \end{aligned}$$

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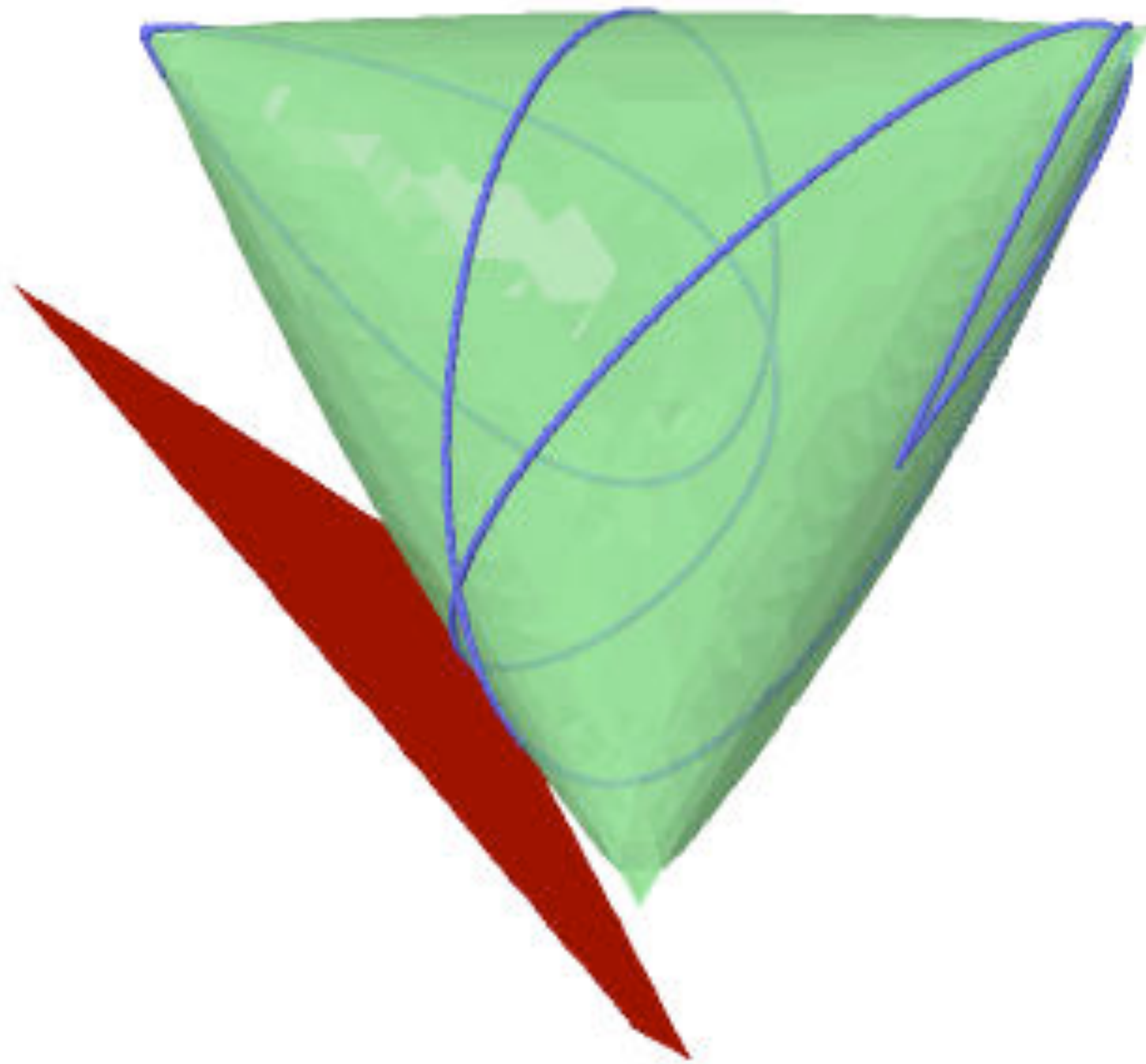

First SDP relaxation - standard!

Solvable in time polynomial in n.

From QCQP to SDP

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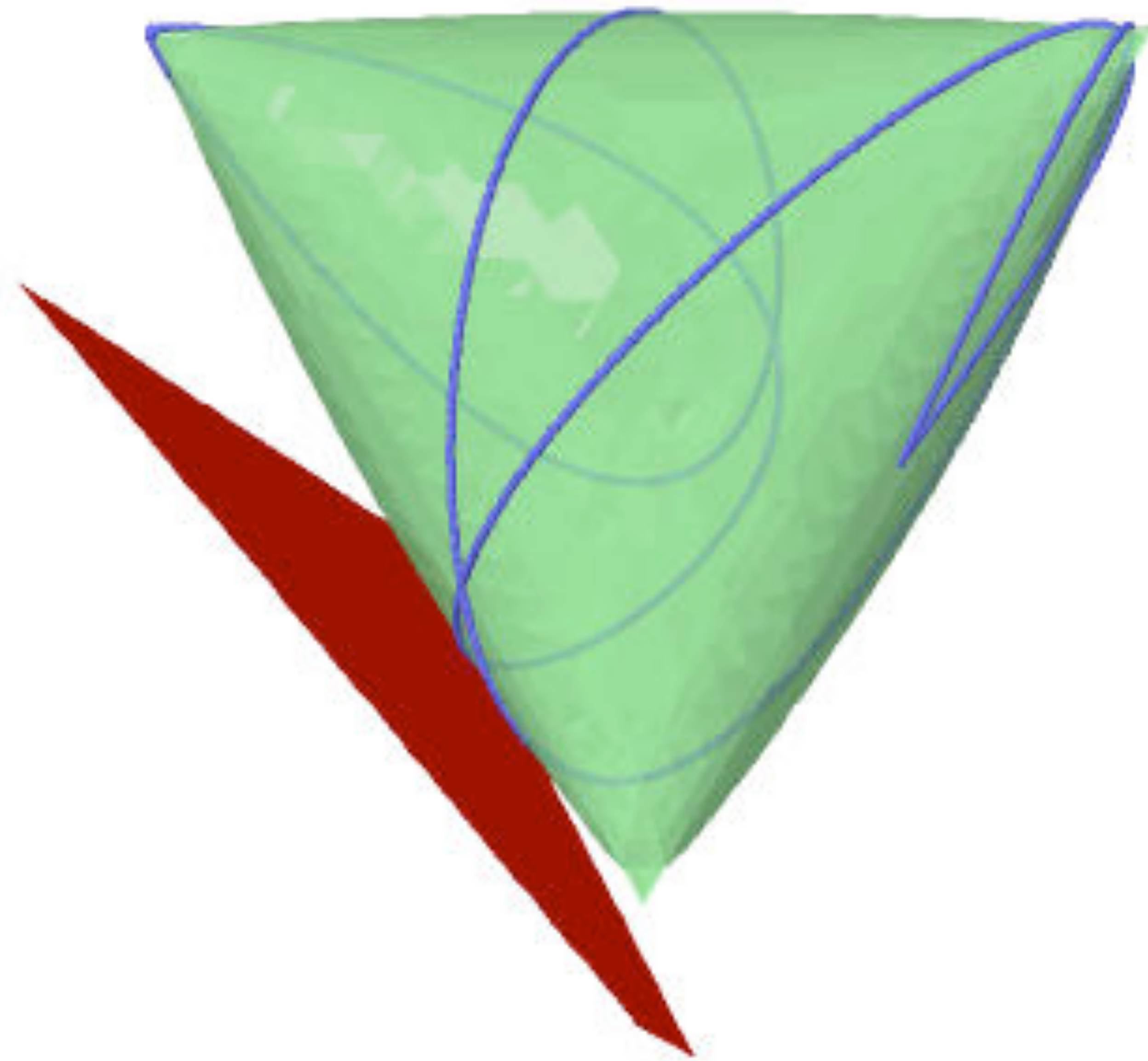
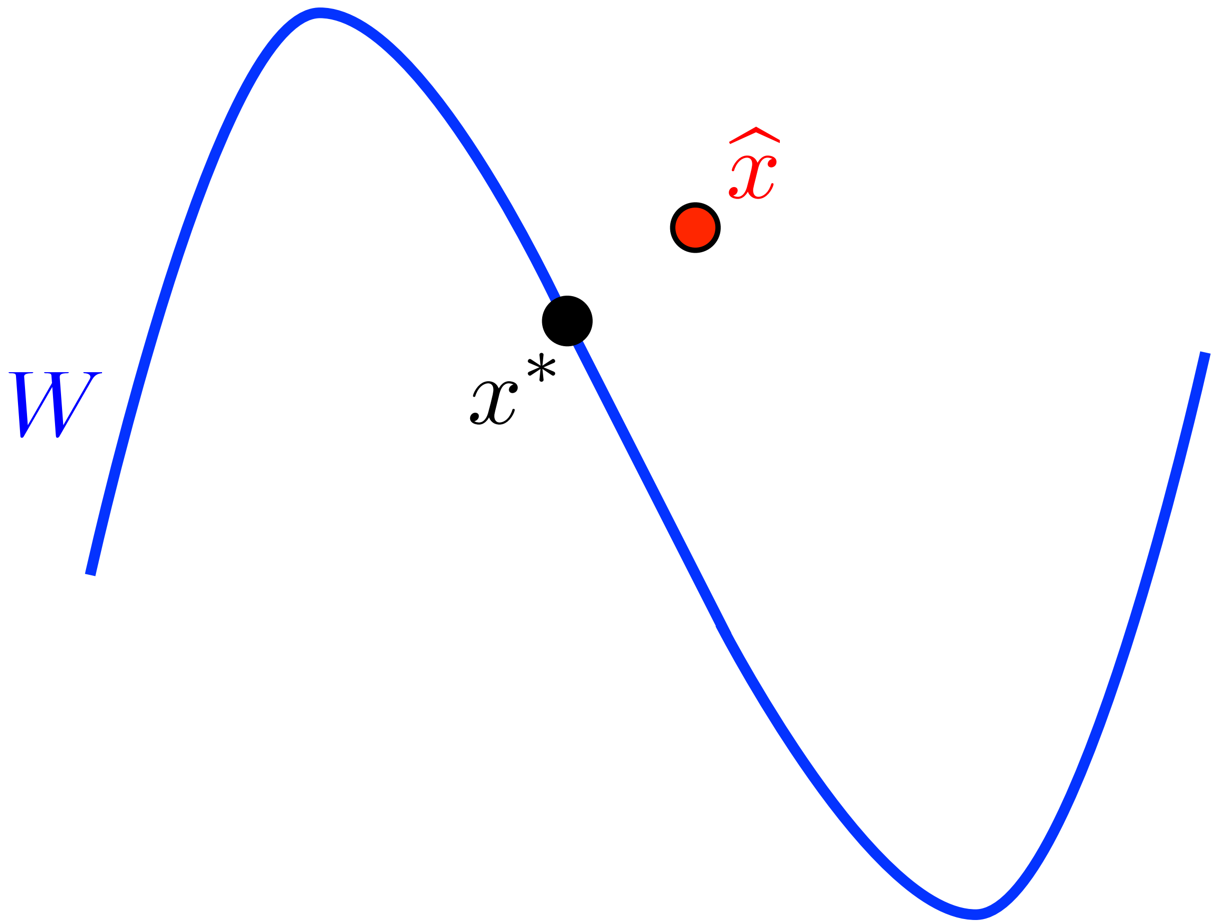
$$\min_{Y} \quad \langle G, Y \rangle \quad \text{s.t.} \quad \langle F_i, Y \rangle = b_i, \quad Y \succeq 0$$

$$\rightarrow Y^* = \begin{bmatrix} \star & y^* \\ (y^*)^\top & \star \end{bmatrix}$$

Candidate solution for QCQP



When Does QCQP = SDP?



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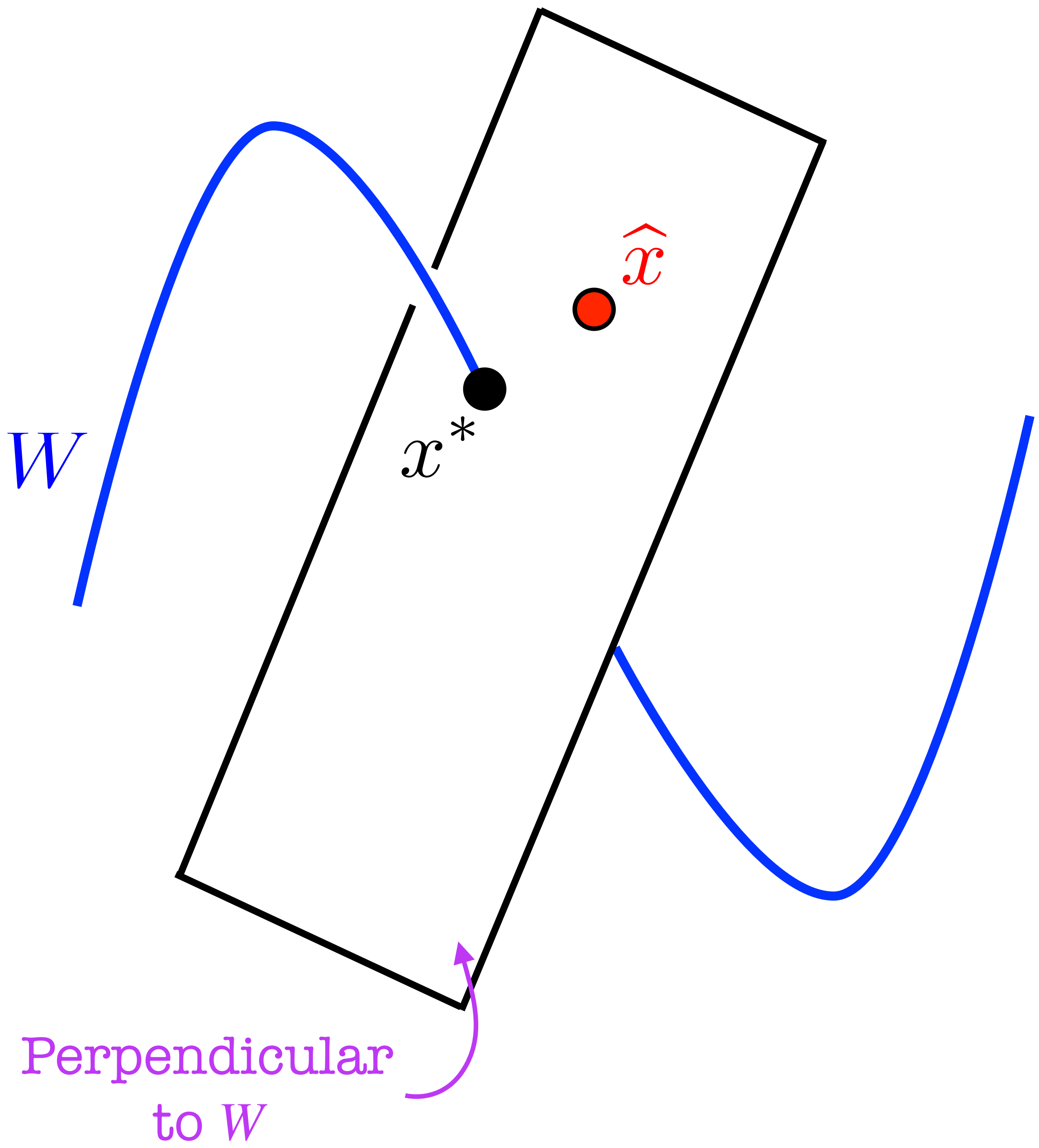

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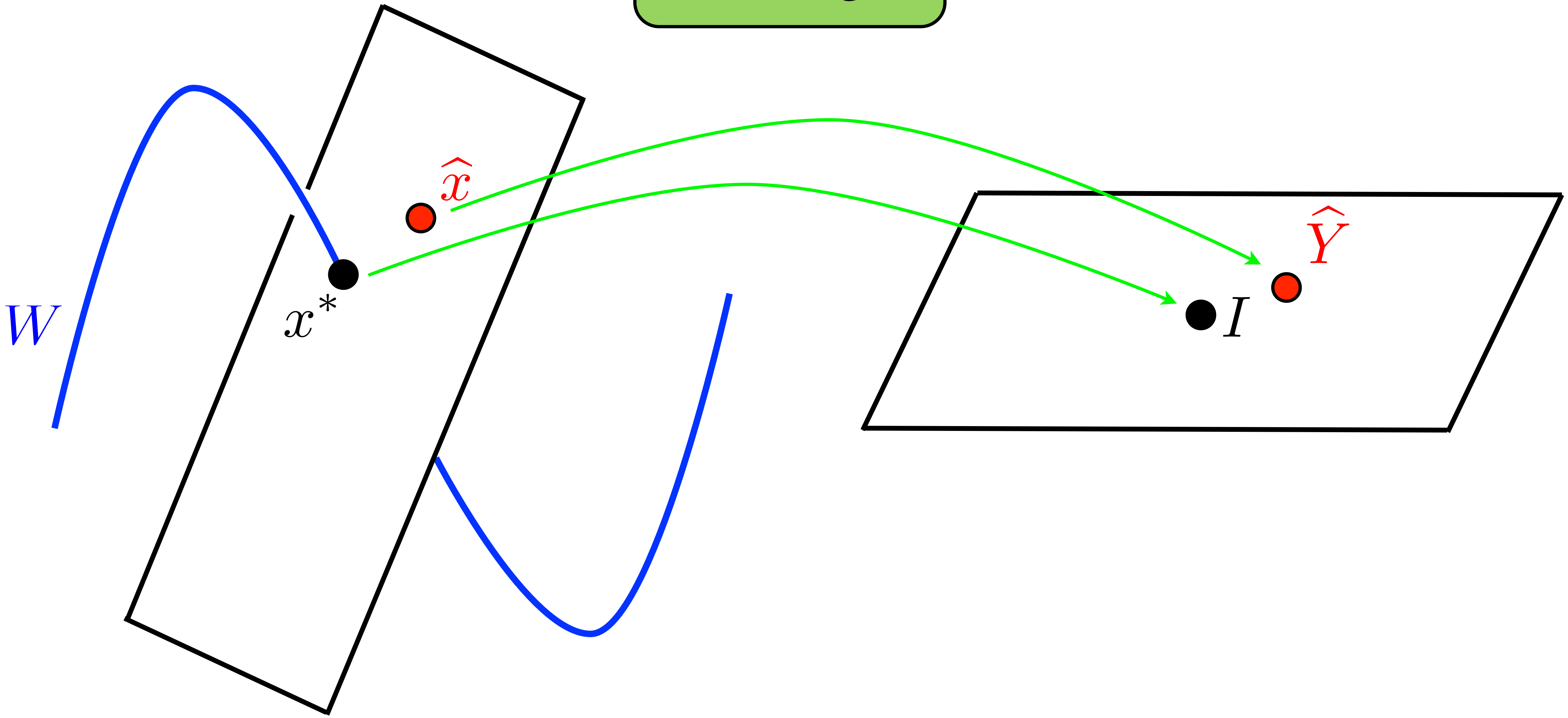
$\mathbb{R}^{2n} \longrightarrow \mathcal{S}^{2n}$



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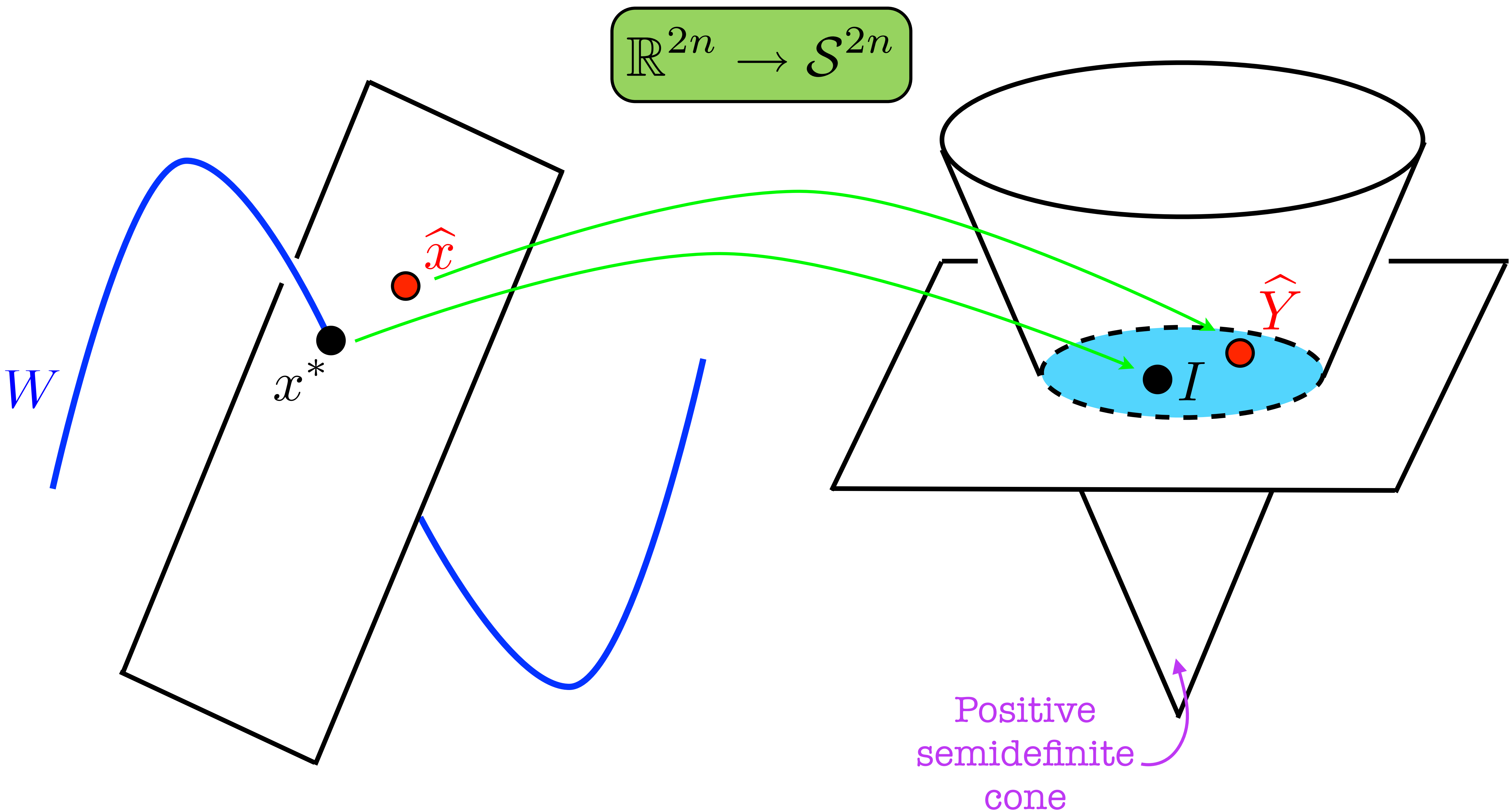

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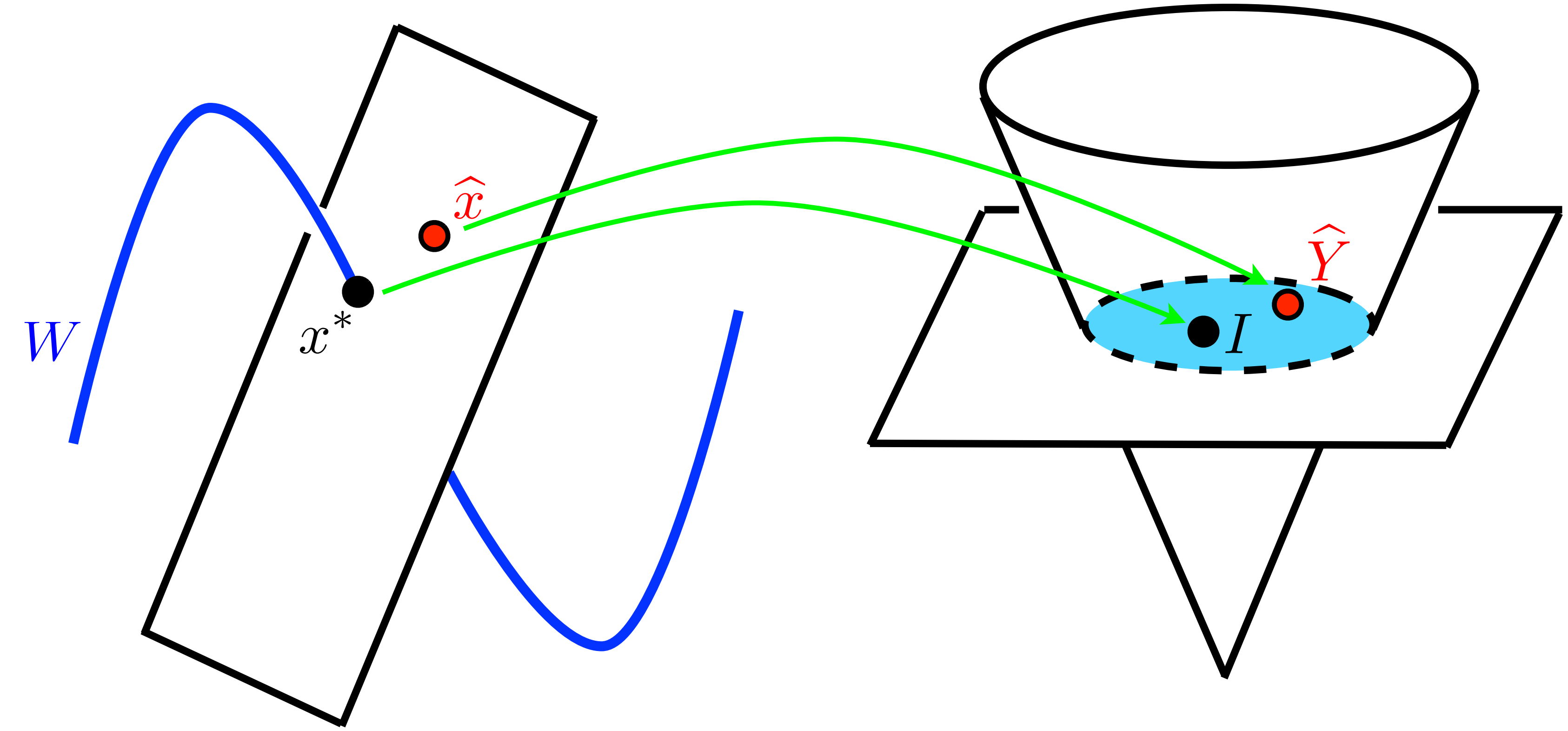
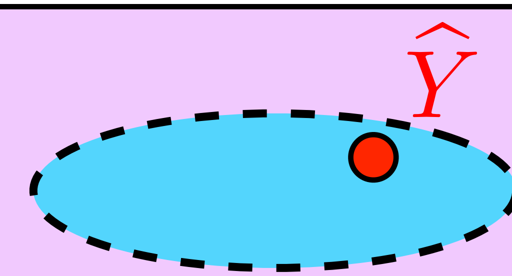
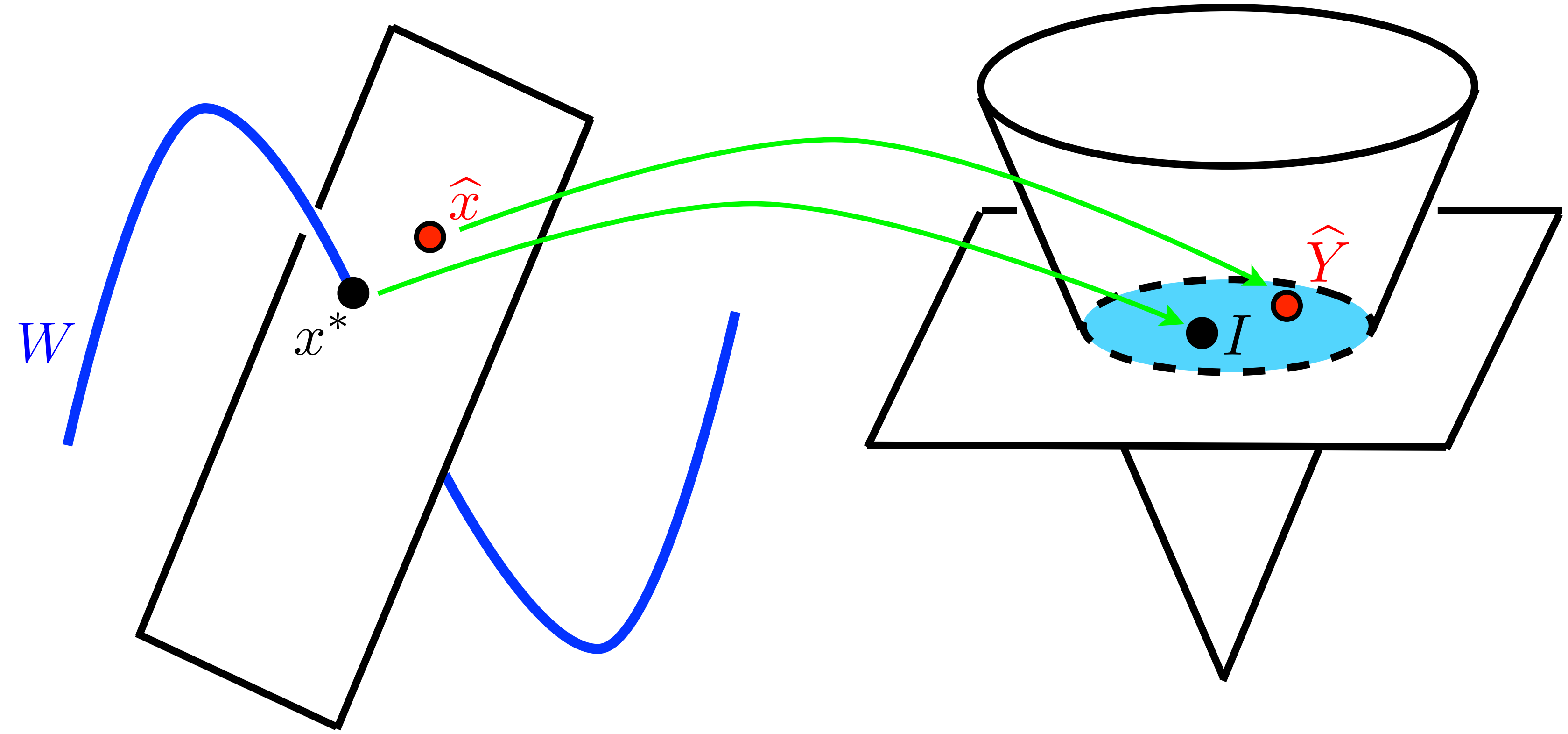


Theorem (A, Agarwal, Thomas)

QCQP = SDP if and only if 

- Can be formulated for general QCQPs with equality constraints.
- Extends previous results in special cases of QCQPs.
- Polynomial time test for optimality.

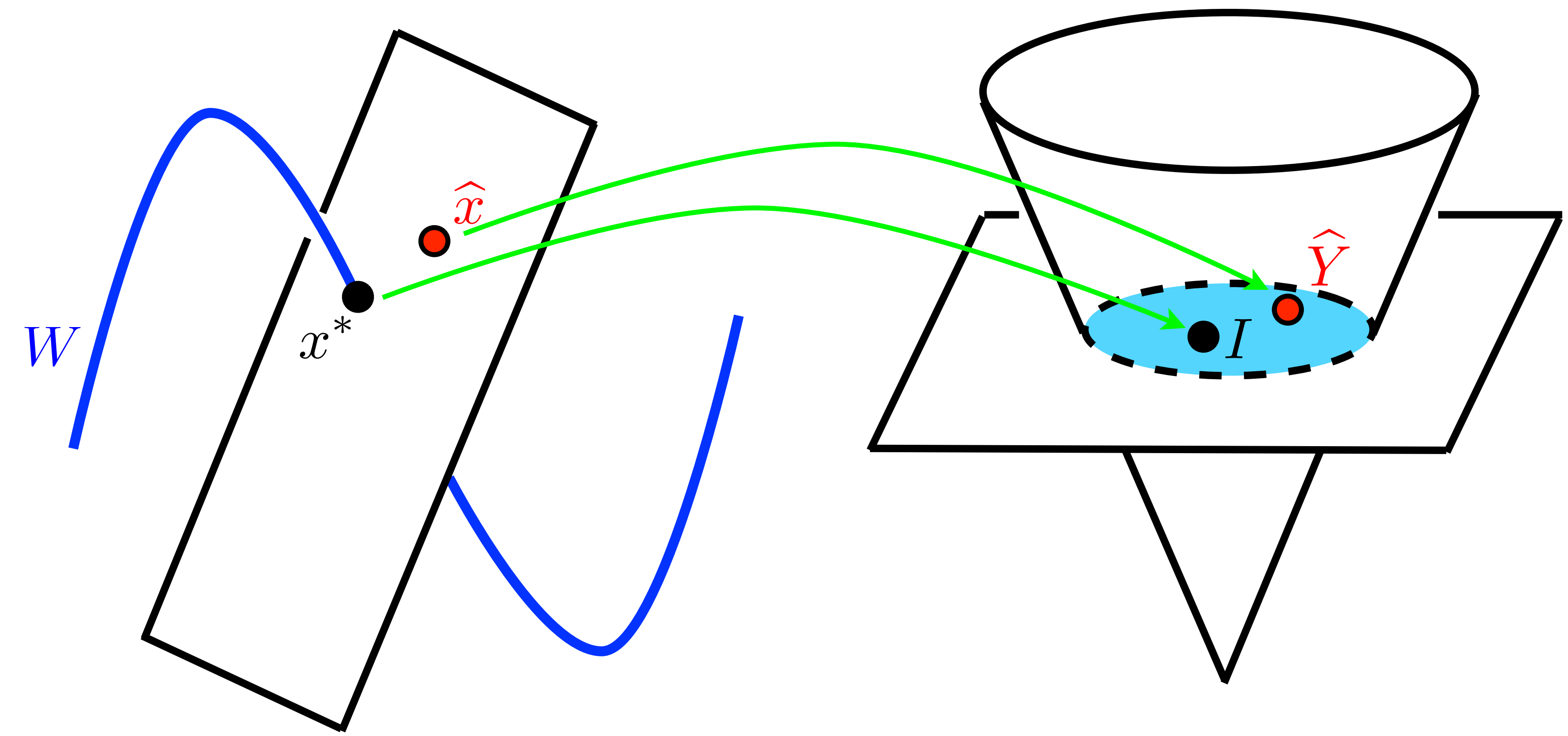
When Does QCQP = SDP?



Theorem (A, Agarwal, Thomas)

Always works in the case of small noise.

When Does QCQP = SDP?



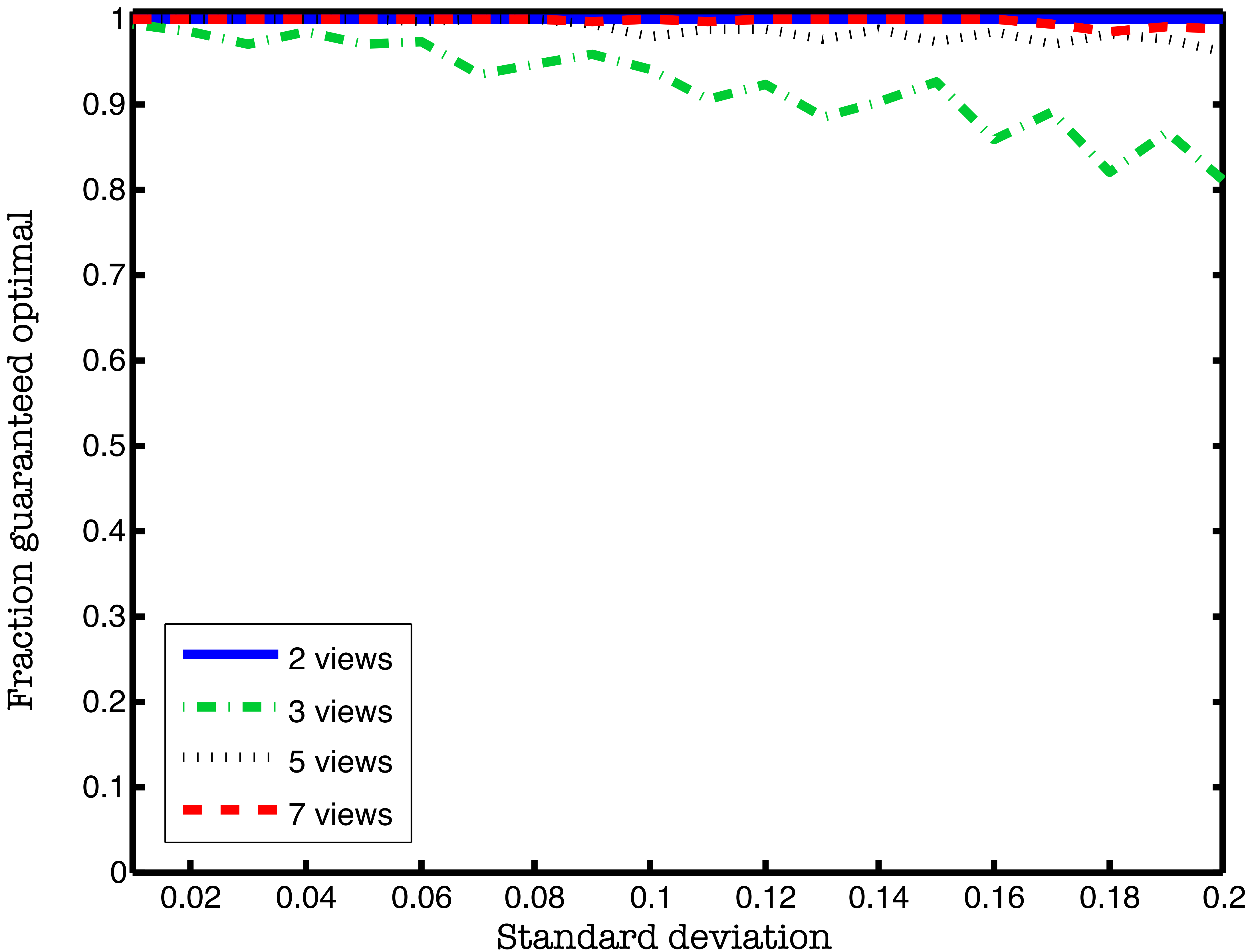
Theorem (A, Agarwal, Thomas)

Always works in the case of small noise.

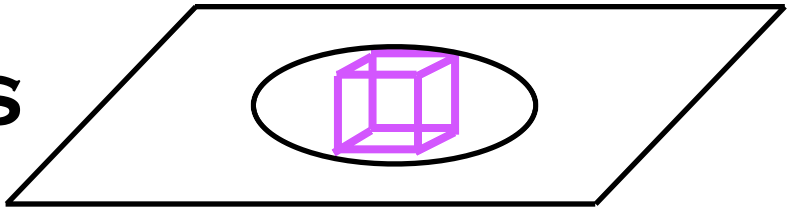
Bonus: Always works for two-view triangulation.

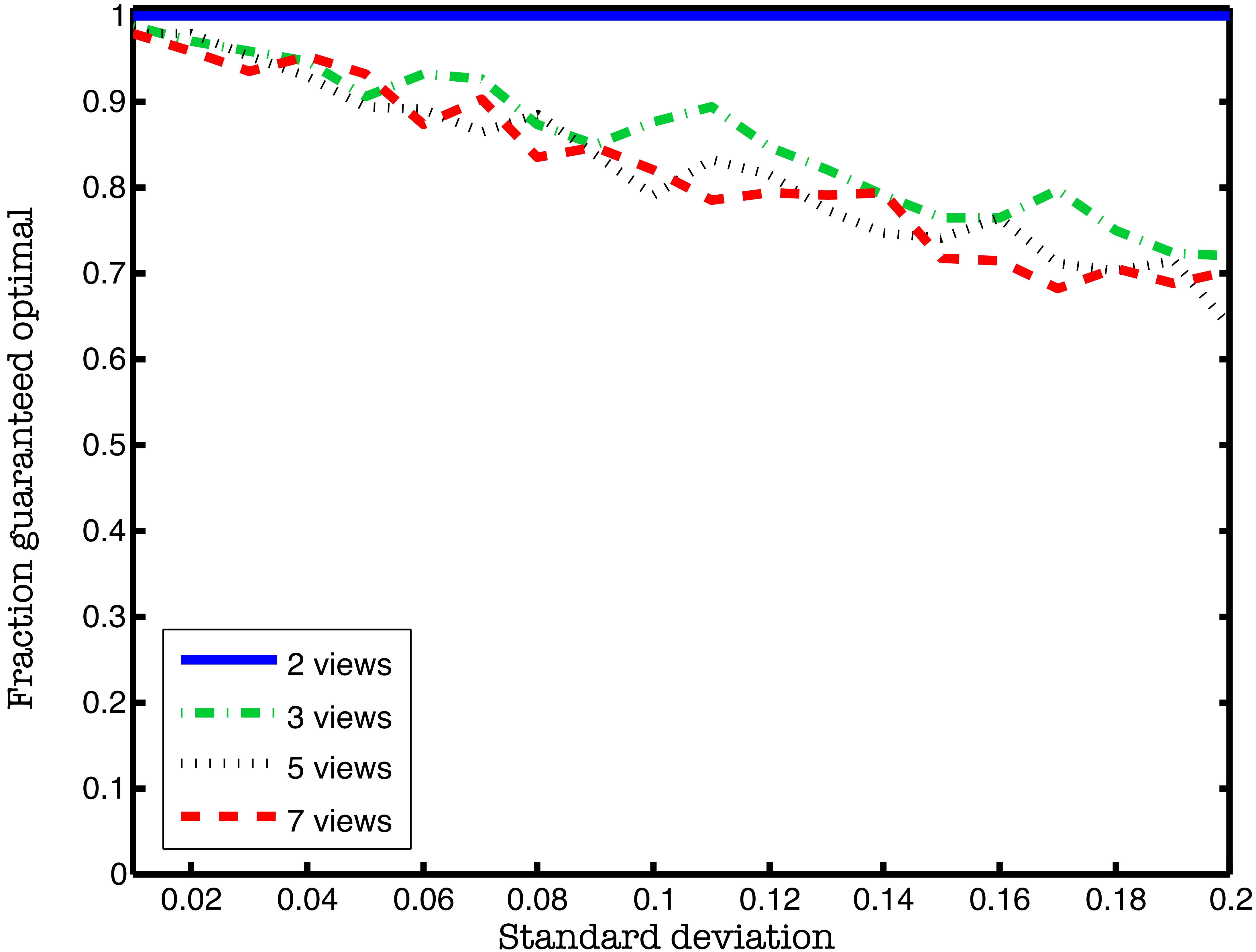
[Moré 93]

Synthetic - Cameras on Sphere

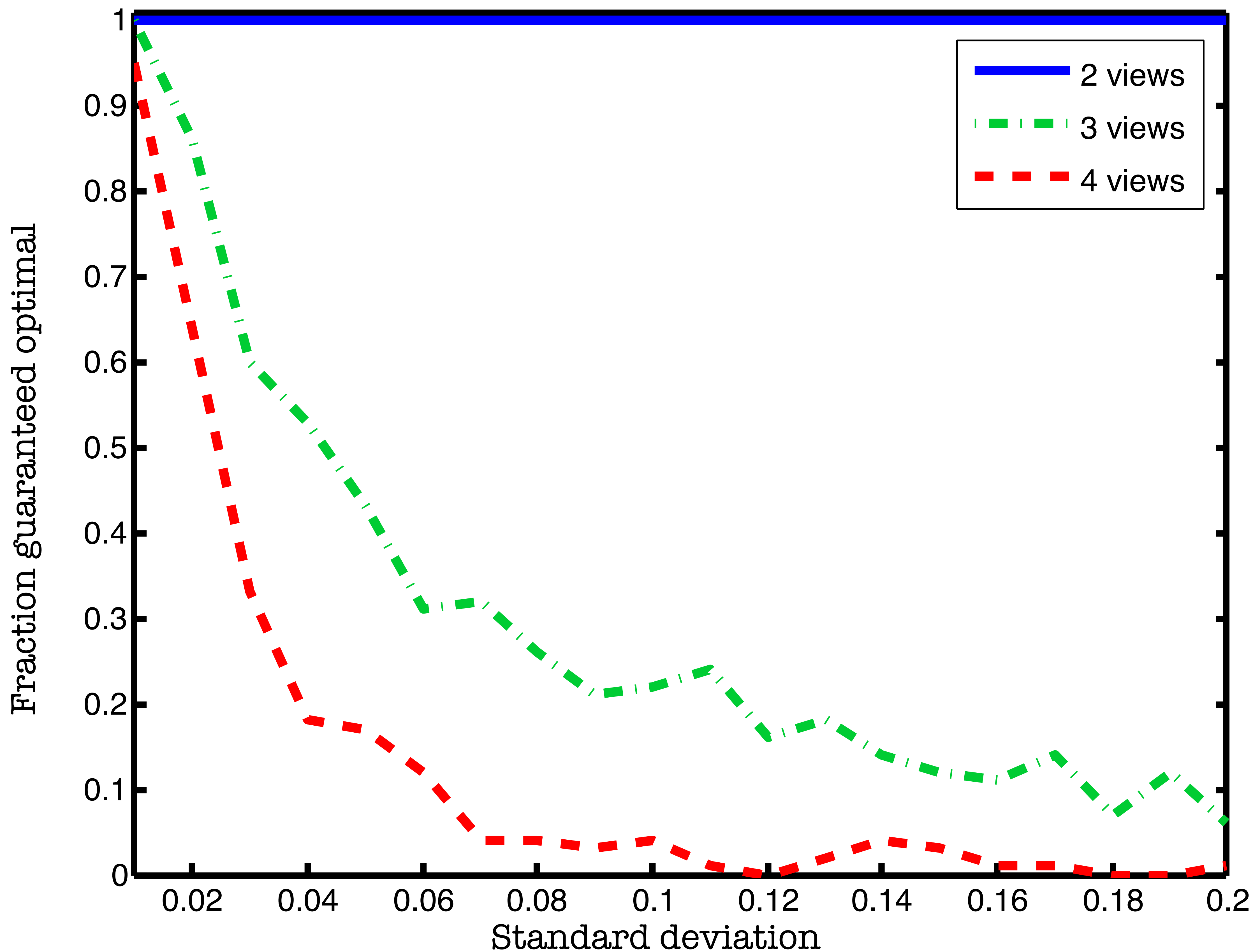


Synthetic - Coplanar Cameras





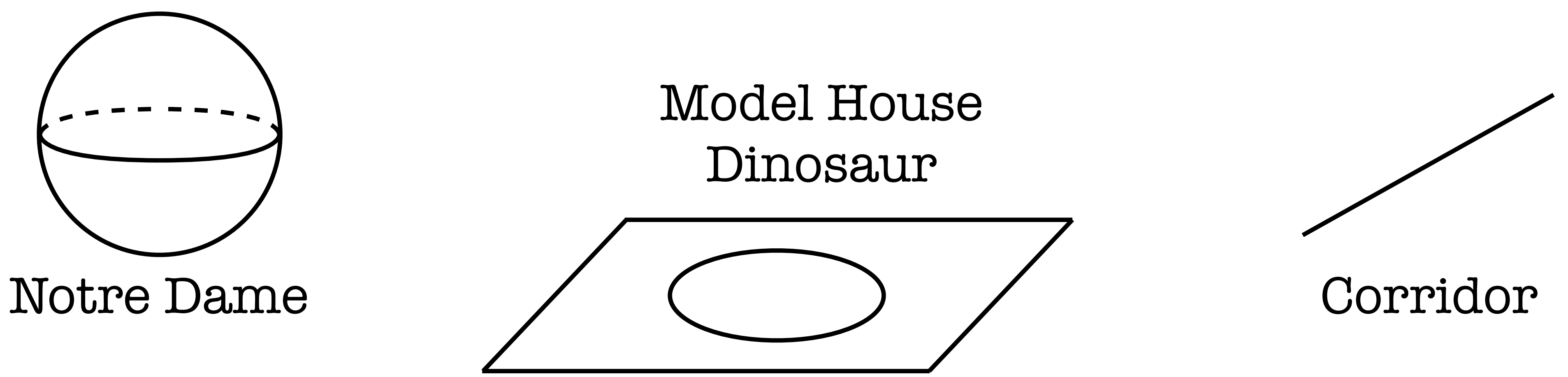
Synthetic - Collinear Cameras



Real Data

Data set	# images	# points	Optimal	Time (sec)
Model House	10	672	100%	143
Corridor	11	737	99.86%	193
Dinosaur	36	4983	100%	960
Notre Dame	48	16,288	98.4%	7200

All the camera configurations are accounted for:



Summary of Our Contributions

- Nearly polynomial time algorithm for triangulation.
- Geometric understanding from constraints.
- General theorems for SDP relaxations of QCQPs.