BAYESIAN NUMERICAL ANALYSIS

CONSIDER: $f(x) = e^{\cosh\left(\frac{x^2+5}{\cos^2(x)+3}\right)}$

WHAT IS $\int_0^1 f(x)\,dx$?

WHAT DOES IT MEAN TO "KNOW A FUNCTION"?

IDEA: ADMIT DON'T KNOW f, PUT A PRIOR ON f (?!)

SAY $f(t) \simeq A + C\,B(t)$ (BROWNIAN MOTION)

OBSERVE: $y_1 = f(x_1), \cdots\ y_n = f(x_n)$

$f \mid y \sim A + C\,B(t)$ CONDITIONED $= y_i$ AT x_i

BAYES RULE FOR f IS STRAIGHT LINE INTEPOLENT

$\hat{I}$ IS TRAPAZOID RULE

<u>WAIT</u> THIS f IS SMOOTH; B.M. NOT <u>MY</u> PRIOR!

OK, USE $f(t) \sim \int_0^t B(s)\,ds$, GET CUBIC SPLINES

OR, INTEGRAT k TIMES; $2k+1$ SPLINES

MAYBE THIS BAYESIAN APPROACH IS NOT SO CRAZY

<u>PROGRAM</u>: TAKE STANDARD NUMERICAL TASKS
(QUADRITURE, INTERPOLATION, ROOTS, MAXIMA, ⋯)
AND SEE IF THEY HAVE BAYESIAN INTERPRETATION

GOOD

- POSSIBLE IN HIGH DIMENSIONS
- GIVES POSTERIOR, NOT JUST BAYES RULE
- GIVES A DIFFERENT WAY OF THINKING ABOUT N. A. (VERSUS 'k^{th} ORDER', '# OPERATIONS')

QUESTIONS

- IS SIMPSONS RULE BAYES? (NOPE)

- IS B.M. THE ONLY PRIOR GIVING TRAPAZOID RULE?
  - POISSON PROCESS (ANY INDEPENDENT INCREMENT)
  - $C\,B(t)$ PREDICTS SAME, $C \perp B(t)$

THEOREM (Williams) IF X_t ON $[0,1]$ HAS CONTINUOUS SAMPLE PATHS AND PREDICTS LIKE B.M., THEN

$$X_t \sim C\,B(t) \qquad C \perp B(t)$$

WHY X_t IS A MARTINGALE
USE LEVY'S CHARACTERIZATION

HISTORY

H. POINCARÉ (1896)

$f(x)$ UNKNOWN ON $\mathbb{R}$

GIVEN $f(x_i) = y_i \quad 1 \le i \le N$

ESTIMATE $f(x)$ (INTERPOLATION)

<u>IDEA</u> ASSUME $f(x) = \sum_{j=0}^{\infty} A_j x^j$

$A_j \sim \mathcal{N}(0, \sigma_j^2)$ SO $f(x)$ IS GAUSSIAN

<u>SO</u> $f(x)|_{y_1 \cdots y_N}$ IS GAUSSIAN

IF $\sigma(x) = \sum_{j=0}^{\infty} \sigma_j^2 x^j$

$\hat{f}(x) = a_1 \sigma(x_1 x) + \cdots + a_N \sigma(x_N x)$

a_i CHOSEN SO $\hat{f}(x_i) = y_i$

"I USE THE METHOD OF CAUSES"

WHATS NEW?

LOTS OF APPLICATIONS IN FUNCTION SPACE SETTING

A. STUART (2010) INVERSE PROBLEMS: A BAYESIAN APPROACH. ACTA NUMERICA

<u>OBSERVE</u> $y = g(u) + \epsilon$

<u>PROBLEM</u> ESTIMATE u

HERE y AND u ARE FUNCTIONS

<u>EX</u> (STOKES EQUATION)

$$\frac{\partial v}{\partial t} = \nu \Delta v - \nabla p + f, \quad (x,t) \in D \times [0,\infty)$$

$$\nabla \cdot v = 0, \qquad v = u \text{ AT } (x,0)$$

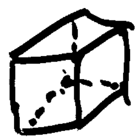
OBSERVE TRACES $z_j(t) = z_j(0) + \int_0^t v(z_j(s), s)\, ds$

AT t_i $1 \le i \le I$, $1 \le j \le J$ THIS IS y

<u>OBJECT</u> ESTIMATE u GIVEN $z_j(t_i)$

LOTS OF USEFUL NEW PRIORS

<u>PBLM</u> GIVEN A SET $\mathcal{X}$, FIND $\left\{\begin{array}{l}\text{USEFUL}\\ \text{NATURAL}\end{array}\right.$ PRIORS INDEXED BY $\mathcal{X}$

<u>EX</u> $\mathcal{X} = C_2^d \leftrightarrow$ 

$x \in \mathcal{X}$, $f(x)$ MEASURES 'THROUGHPUT'

FIND $\max_{x \in \mathcal{X}} f(x)$

WITH A PRIOR, CAN CHOOSE INFORMATIVE OBSERVATIONS SEQUENTIALLY

<u>IDEA</u> (YLVISAKER) USE DYNKINS ISOMORPHISM:

EASY TO MAKE (REVERSIBLE) MARKOV CHAINS $\pi(x)$, $K(x,y)$ $\left(\pi(x)K(x,y) = \pi(y)K(y,x)\right)$ INDEXED BY $\mathcal{X}$

$$G(x,y) = \sum_{n=0}^{\infty} K^n(x,y) \quad \text{(NEED } K \text{ ABSORBING)}$$

$G(x,y)$ IS P.D. SO THE COVARIANCE OF A GAUSSIAN FIELD

¿ SO WHAT ?

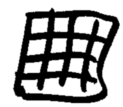
$$E(X_x \mid X_s, s \in S) = \sum_{s \in S} c_s X_s$$

$$c_s = P\{ X_y, \text{ STARTED AT } x, \text{ FIRST HITS } S \text{ IN } s \}$$

POINT, $c_s \geq 0$, COVARIANCES POSITIVE

THERE IS A CONVERSE

⋮

POINT: IF THE UNDERLYING MARKOV CHAIN IS
N.N. RANDOM WALK ON A GRAPH 

THEN X_x IS THE GAUSSIAN FREE FIELD

AN OBJECT OF INTENSE STUDY IN MODERN PROBABILITY

POINT: THERE ARE NOW GAUSSIAN ANALOGS

⋮

STEVE EVENS AND I HAVE A DUAL CONSTRUCTION WHICH GIVES NEGETIVE COVARIANCES (AND ALL SORTS OF NICE PROPERTIES):

- LET $\pi(x), K(x,y)$ BE REVERSIBLE ON (FINITE) $\mathcal{X}$.
- FORM A GRAPH ON $\mathcal{X}$ WITH $x \sim y \iff K(x,y) > 0$
- CHOOSE AN ORIENTATION $\epsilon_{xy} = \pm 1$ EACH EDGE
- PUT A MEAN 0, VAR $\pi(x)K(x,y)$ GAUSSIAN ON EACH EDGE $Z(x,y)$
- SET $Z_x = \sum_y \epsilon(x,y) Z_{xy}$
- THEN $\{Z_x\}_{x \in \mathcal{X}}$ HAS COVARIANCES $\Sigma(x,y) = \begin{cases} \pi(x) & x = y \\ -\pi(x)K(x,y) & - \end{cases}$
- IF $K(x,y) = 0$ $\mathrm{COV}(Z_x, Z_y) = 0$

$$Z_1 = Z_{51} - Z_{12},$$
$$Z_2 = Z_{12} - Z_{23}$$
$$\vdots$$

WITH KHARE, WE HAVE SIMILAR PROCESS WITH GENERAL SGN PATTERN, ON GENERAL SPACES

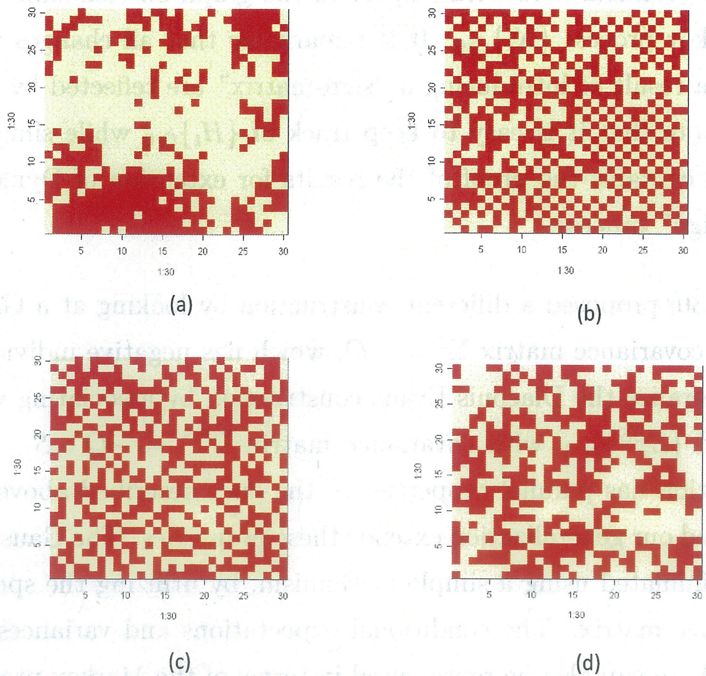


Figure 2.1: Signs of realizations of various Gaussian fields for a 30×30 grid (with a different sign matrix) (a) Dynkin's construction (b) Generalized Dynkin's construction (c) Diaconis-Evans' construction (d) Generalized Diaconis-Evans' construction.

QUESTIONS

- IS THERE A PRIOR GIVING EVEN ORDER SPLINES (THE HALF INTEGRAL DOESN'T WORK)

- CHARACTERIZE $f(t) \sim \int_0^t B(s)\,ds$ A LA HARNESS'S

- MORE GENERALLY, WHEN DOES THE BAYES RULE CHARACTERIZE THE PRIOR?

- DESIGN: USE PRIOR TO SAY WHERE TO TAKE OBSERVATIONS.

  eg. $f(t) \sim B(t)$ ON $[0,1]$ TO ESTIMATE $I = \int_0^1 f(s)\,ds$.

  BEST n POINTS ARE $\frac{2i}{2n+1}$ $1 \le i \le n$

  eg $n=2$, $\frac{2}{5}, \frac{4}{5}$

- TRY THESE THINGS OUT (ITS EASIER TO PROVE THEOREMS THAN TO DO SERIOUS EXAMPLES)