Ricci Flow Manifold Embedding

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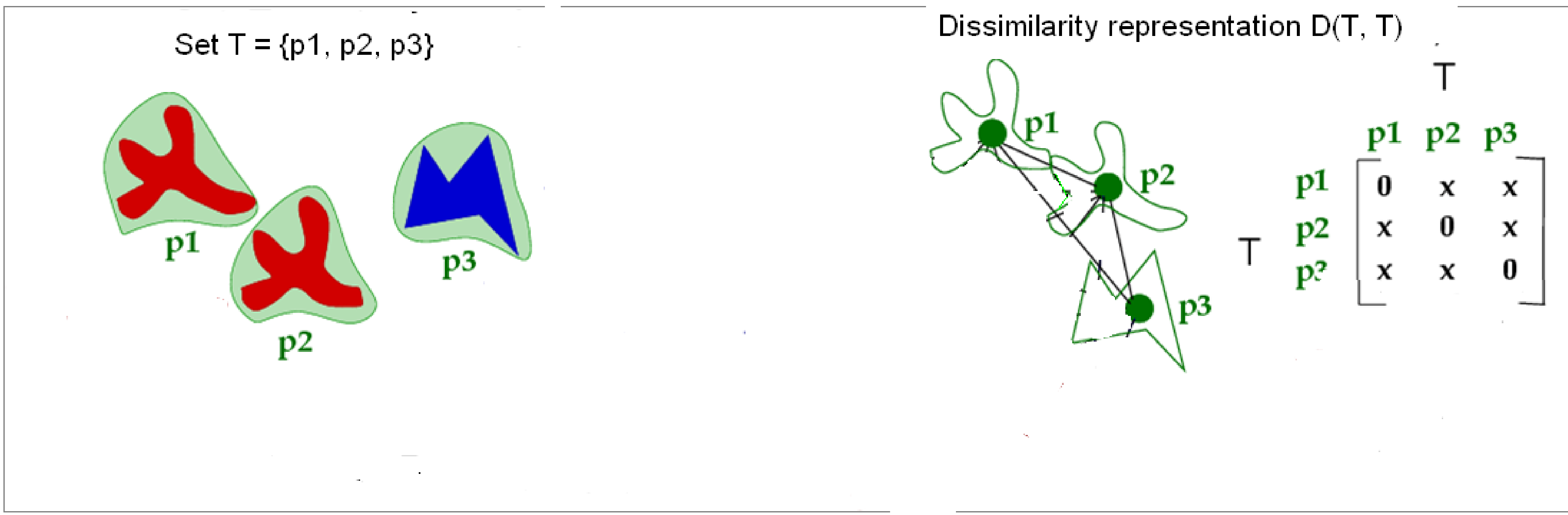
Representation of objects

– Feature-based vector representation

Raw data → determine n features → objects as points in n dimensional space

– Dissimilarity representation



indefinite dissimilarity data

- The kernel embedding only works if **K** is a Mercer kernel, **K** must be positive semidefinite (zero or positive eigenvalues)

$$\mathbf{K} = \mathbf{U}\Lambda\mathbf{U}^T, \mathbf{X} \neq \mathbf{U}\Lambda^{1/2} \quad \text{if} \;\; \lambda_i < 0$$

- Otherwise the dissimilarities are **indefinite (non-Euclidean)**

Related embedding work

- Multi-dimensioanl scaling(MDS) (Torgerson, 1958)
- Isomap (Silva, Saul and Lanford, 2000)
- Locally linear embedding (LLE) (Roweis and Saul, 2000)
- Laplacian eigenmap embedding (Belkin and Niyogi, 2002)
- The pseudo Euclidean space embedding (Goldfarb, 1984)
- The positive part of the pseudo Euclidean space embedding (Kernel embedding)
- Dissimilarity Enlargement by a Constant (Pekalska, Duin, Gunter and Bunke, 2004)

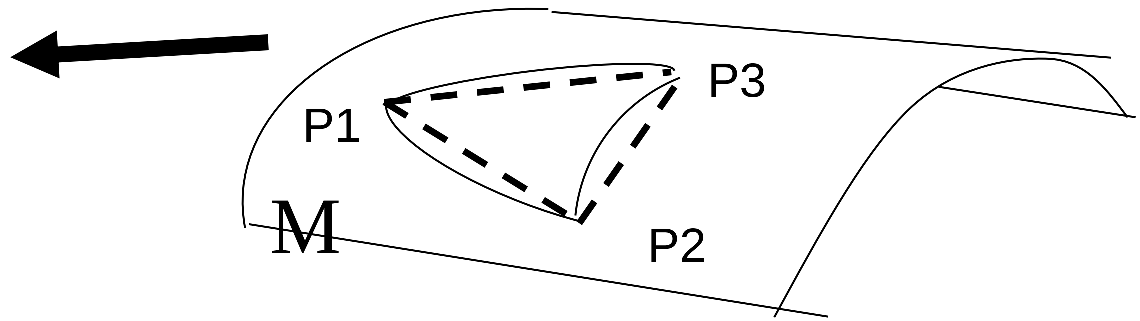
What can we do to correct non-Euclidean effects

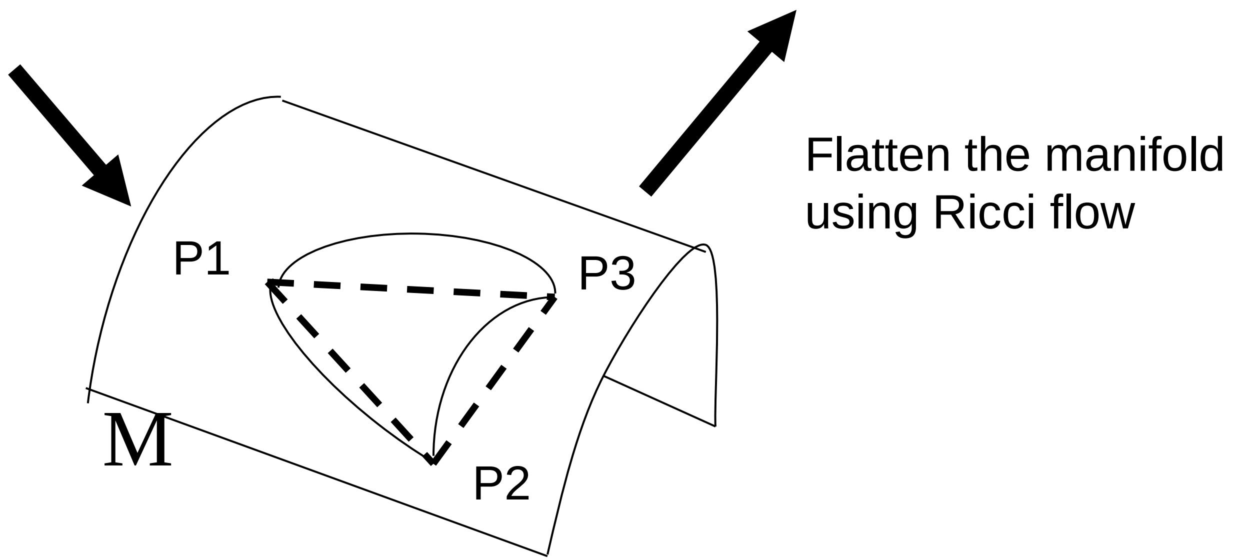
- Riemennian space
- Space is curved – distances are not Euclidean, indefinite similarities
- Distances are measured by geodesics

- Euclidean space
- Flat, curvatures are zero
- The geodesic distances and the Euclidean distances are equal

	p1	p2	p3
P1	0	x	x
P2	x	0	x
P3	x	x	0

$$D = \begin{bmatrix} 0 & x & x \\ x & 0 & x \\ x & x & 0 \end{bmatrix}$$





How can we flatten the curved manifold

- Ricci flow
- a process evolves a metric in direction of its Ricci curvature
- (Hamilton in 1981)
- geometric evolution equation $$\frac{\mathrm{d}g_{ij}}{\mathrm{d}t} = -2\boldsymbol{K}\, g_{ij}$$

In a constant curvature Riemannian space

$$\frac{dK}{dt} = \begin{cases} -2K^2 & \text{elliptic hypersphere,} \\ 2K^2 & \text{hyperbolic space.} \end{cases}$$

$$d_E(u,v) = \begin{cases} \frac{2}{K^{\frac{1}{2}}} \sinh(\frac{K^{\frac{1}{2}}}{2} d_G(u,v)) \\ \frac{2}{|K|^{\frac{1}{2}}} \sin(\frac{|K|^{\frac{1}{2}}}{2} d_G(u,v)) \\ d_G(u,v) \end{cases}$$

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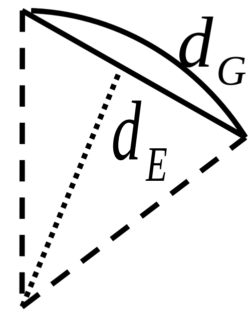
(Lindman and Caelli(1978))

Curvature computation

Newton's method:

$$K_{n+1}^{\frac{1}{2}} = K_n^{\frac{1}{2}} - \frac{K_n^{\frac{1}{2}} d_E - 2\sin\frac{K_n^{\frac{1}{2}}}{2} d_G}{d_E - d_G \cos\frac{K_n^{\frac{1}{2}}}{2} d_G}$$

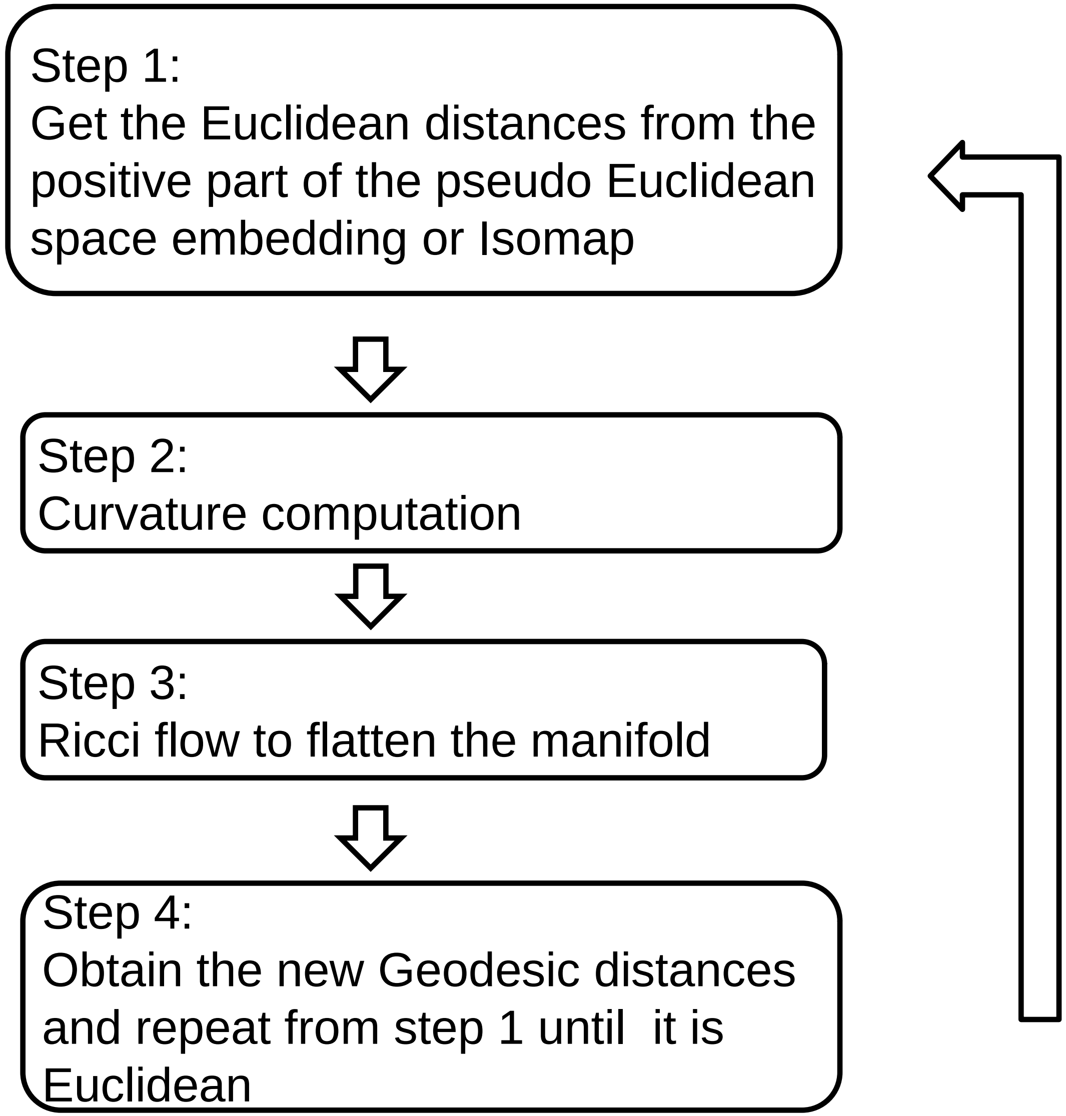


Initial value: use difference between geodesic and Euclidean distance to approximate the Gaussian curvature on the manifold

$$K_0(u,v) = \frac{24(d_G(u,v) - d_E(u,v))}{d_G^3(u,v)}$$

- Updated curvature: $$K = \frac{K_0}{1 \pm 2K_0 t}$$

- # Main steps
- Given non-Euclidean distance matrix:



Problems: instability in the embedding

- The embedding methods affects the magnitude of curvature seperately for each individual edge
- The stronger the curvature, the faster the flattening of the metric
- The positive part of the pseudo Euclidean space embedding preserves the global distances

How can we smooth curvatures over a graph

- Diffusion(Heat) Kernel: create a smoother distribution

$$H(t) = \exp[-Lt]$$

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- The evolution of a function under this kernel
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$$F(t) = H(t)F(0)$$

Regularization step

- Start with the initial Gaussian curvatures

Step 1:
Construct the n nearest graph

⇩

Step 2:
Construct the dual graph

⇩

Step 3:
Obtain the regularised curvature

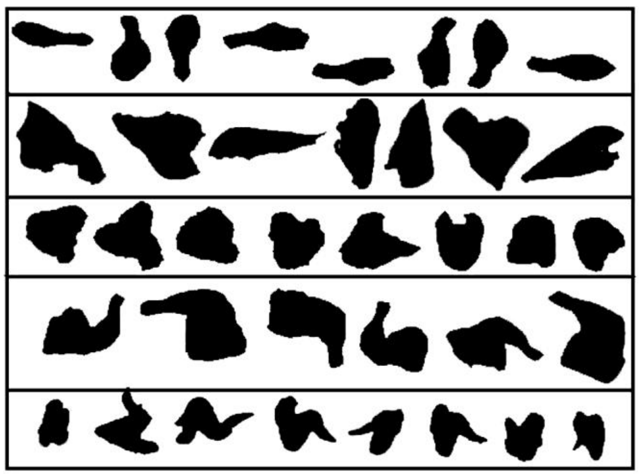
$$\boldsymbol{K}_{reg} = \exp[-\hat{L}t]\boldsymbol{K}$$

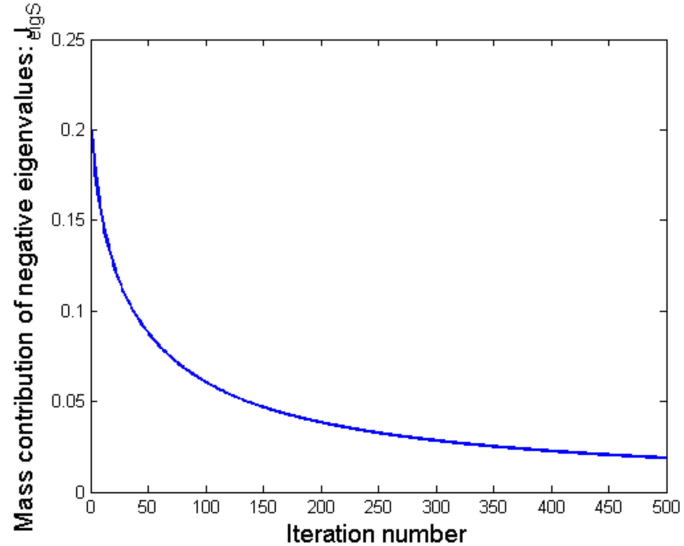
Results

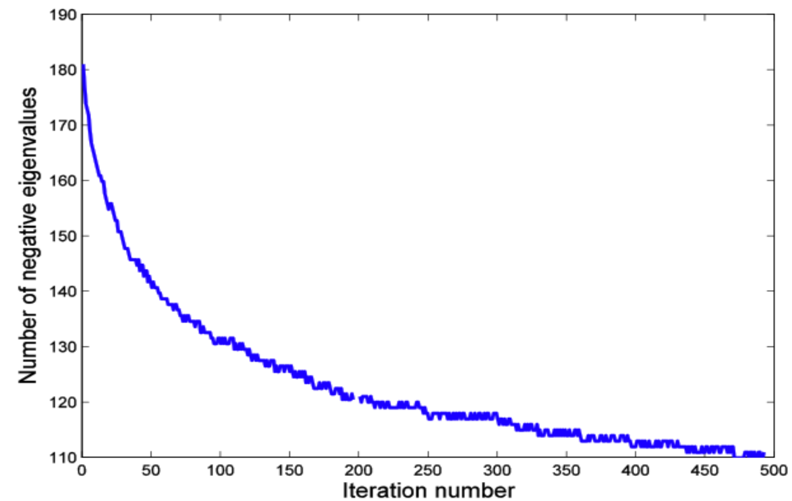
The negative eigenfraction

$$F_{eigS} = \sum_{\lambda_i<0} |\lambda_i| / \sum_{i=1}^{N} |\lambda_i|$$

Chicken pieces dataset





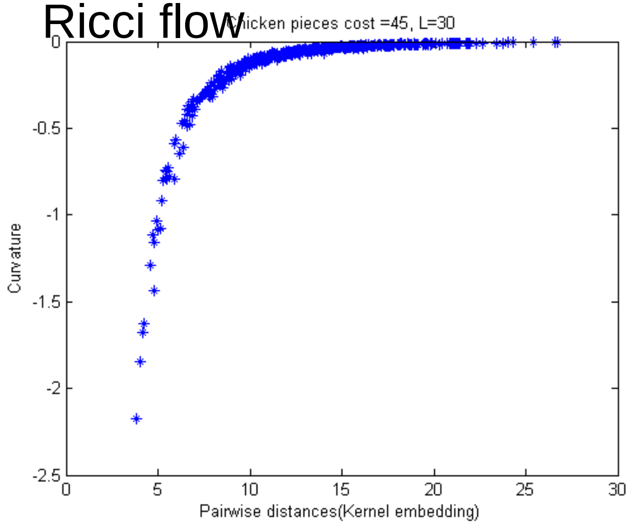


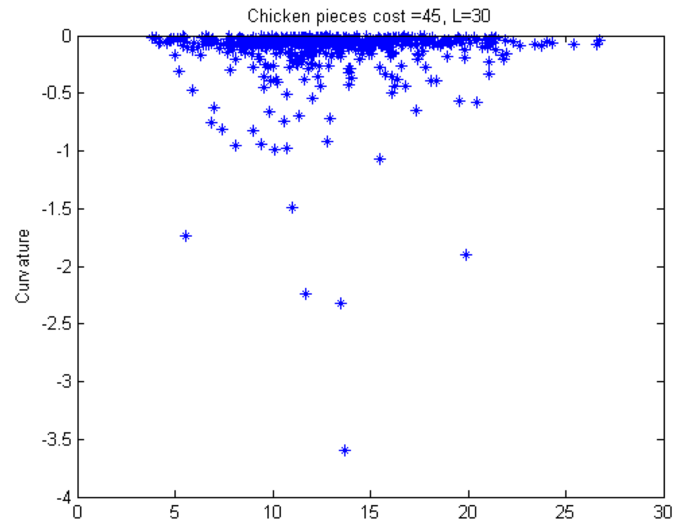
Results

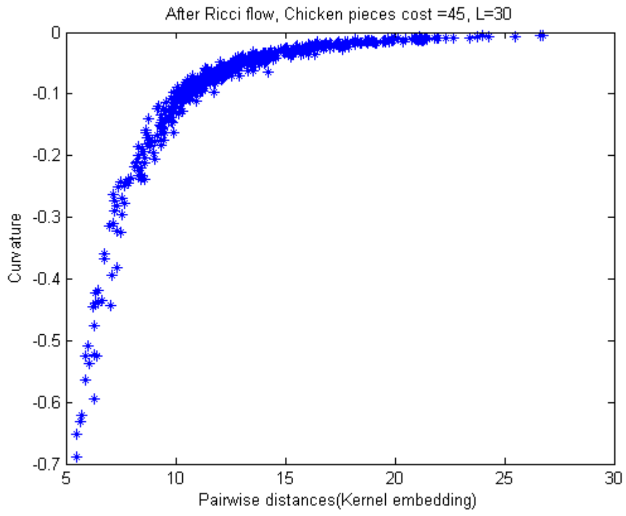
Kernel-embedding Ricci flow

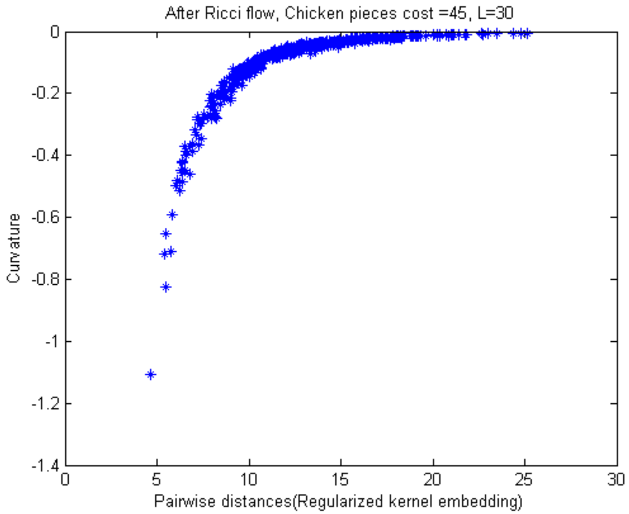
Regularised kernel-embedding

Ricci flow

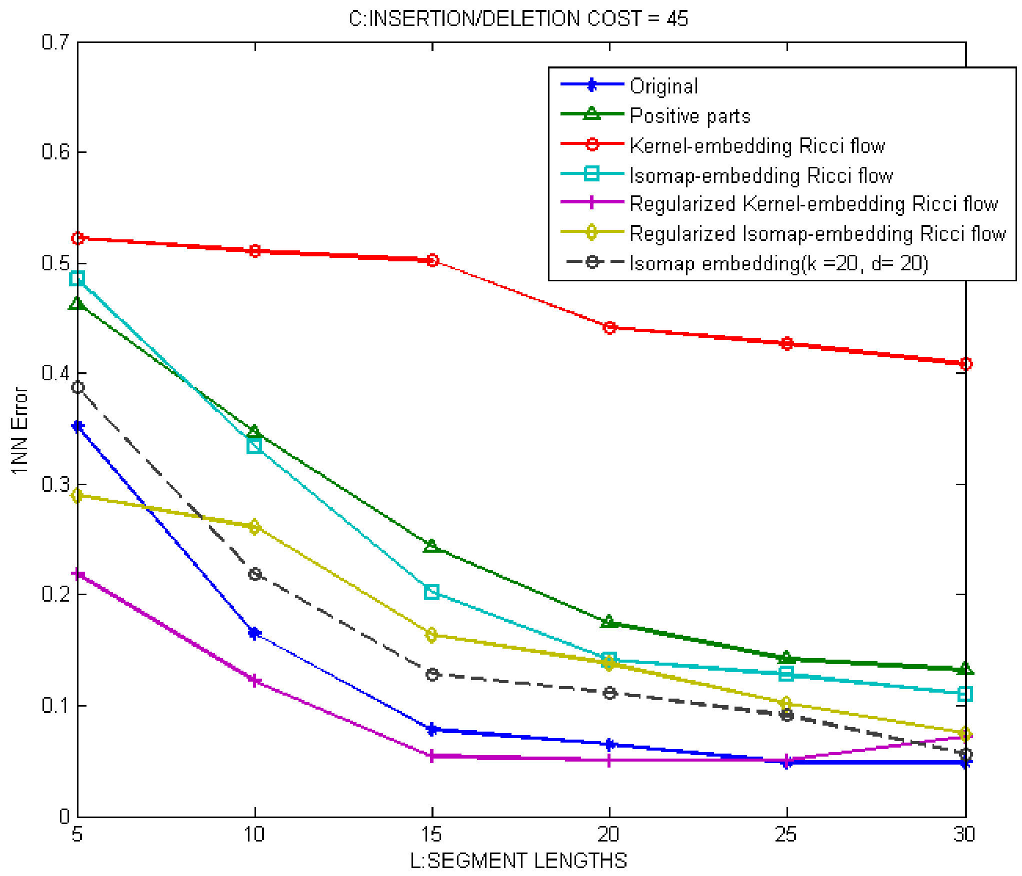








Results



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0.7
0.6
0.5
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1NN Error
5
10
15
20
25
30
L:SEGMENT LENGTHS
Original
Positive parts
Kernel-embedding Ricci flow
Isomap-embedding Ricci flow
Regularized Kernel-embedding Ricci flow
Regularized Isomap-embedding Ricci flow
Isomap embedding(k =20, d= 20)

Conclusion

- Showed methods for evolving a non-Euclidean dissimilarity measure into a form where geometric classification methods can be applied to the data
- The process is unstable due to local fluctuation of curvatures and can be stabilised by regularising the curvatures using Heat kernel

- Future work
- How to reduce the influence of embedding to curvatures?
- How to map new points on the manifold?