Monte-Carlo Planning:
Basic Principles and Recent Progress

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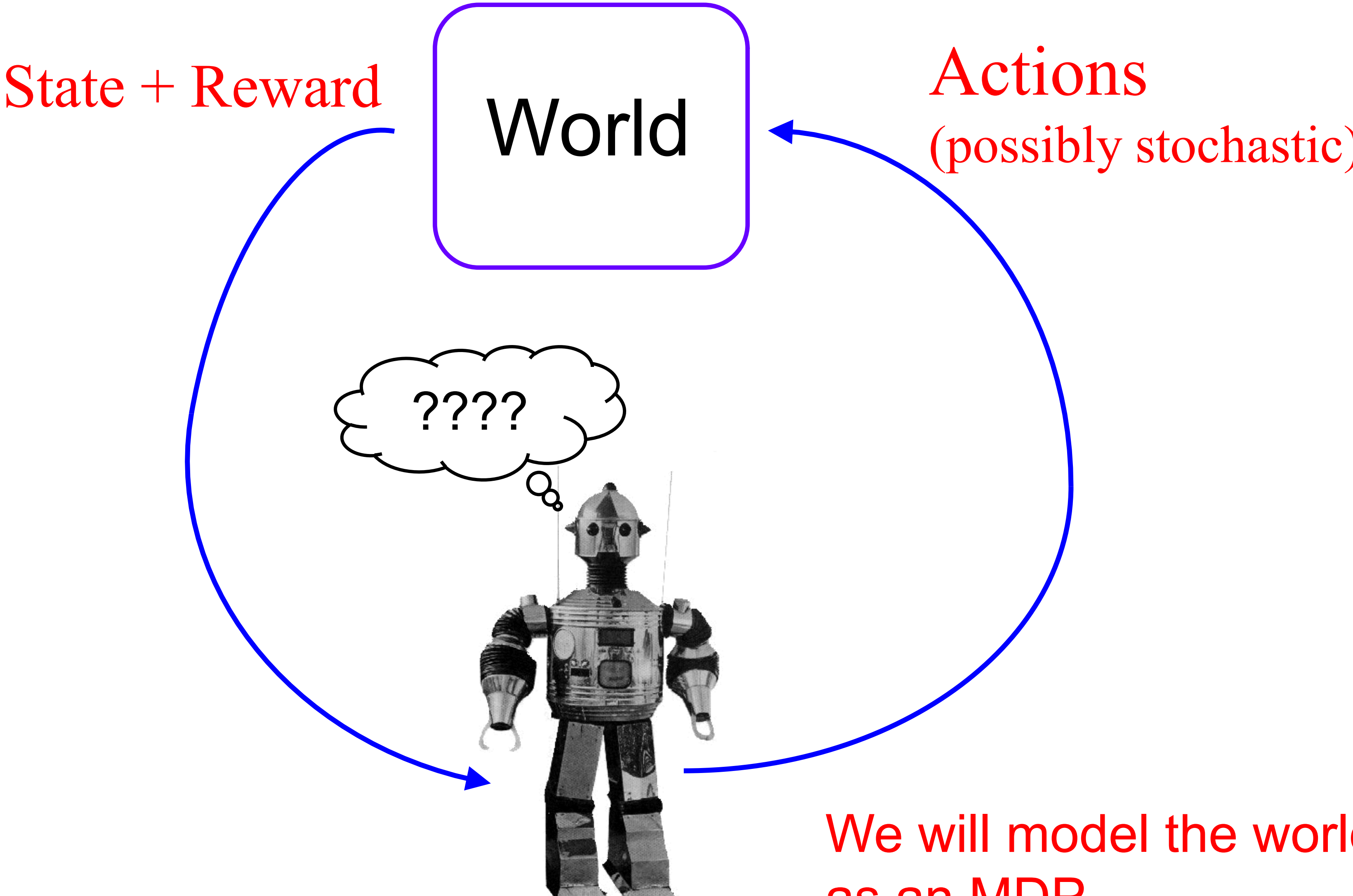
Outline

- Preliminaries: Markov Decision Processes
- What is Monte-Carlo Planning?
- Uniform Monte-Carlo
  - Single State Case (PAC Bandit)
  - Policy rollout
  - Sparse Sampling
- Adaptive Monte-Carlo
  - Single State Case (UCB Bandit)
  - UCT Monte-Carlo Tree Search

Stochastic/Probabilistic Planning: Markov Decision Process (MDP) Model



We will model the world as an MDP.

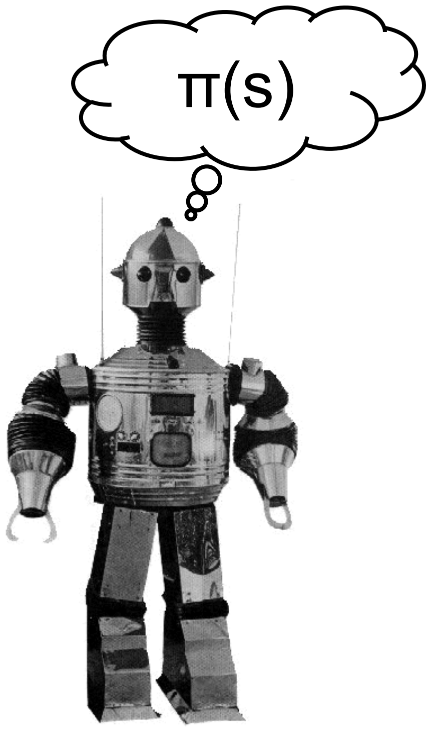
Markov Decision Processes

- An MDP has four components: S, A, P_R, P_T:
  - finite state set S
  - finite action set A
  - Transition distribution $P_T(s' \mid s, a)$
    - Probability of going to state **s'** after taking action **a** in state **s**
    - First-order Markov model
  - Bounded reward distribution $P_R(r \mid s, a)$
    - Probability of receiving immediate reward **r** after taking action **a** in state **s**
    - First-order Markov model

Policies ("plans" for MDPs)

- Given an MDP we wish to compute a policy
  - Could be computed offline or online.

- A policy is a possibly stochastic mapping from states to actions
  - $\pi : S \rightarrow A$
  - $\pi(s)$ is action to do at state s
  - specifies a continuously reactive controller



How to measure goodness of a policy?

Value Function of a Policy

- We consider finite-horizon discounted reward, discount factor $0 \leq \beta < 1$
- $V_\pi(s,h)$ denotes expected h-horizon discounted total reward of policy π at state s
  - Each run of π for h steps produces a random reward sequence: R_1 R_2 R_3 … R_h
  - $V_\pi(s,h)$ is the expected discounted sum of this sequence

$$V_\pi(s,h) = E\left[\sum_{t=0}^{h} \beta^t R_t \mid \pi, s\right]$$

- Optimal policy π* is policy that achieves maximum value across all states

Relation to Infinite Horizon Setting

- Often value function $V_\pi(s)$ is defined over infinite horizons for a discount factor $0 \le \beta < 1$

$$V_\pi(s) = E\,[\sum_{t=0}^{\infty} \beta^t R^t \mid \pi, s\,]$$

- It is easy to show that difference between $V_\pi(s,h)$ and $V_\pi(s)$ shrinks exponentially fast as h grows

$$\left|V_\pi(s) - V_\pi(s,h)\right| \le \left(\frac{R_{\max}}{1-\beta}\right)\beta^h$$

- **h-horizon results apply to infinite horizon setting**

Computing a Policy

- Optimal policy maximizes value at each state
- Optimal policies guaranteed to exist [Howard, 1960]
- When state and action spaces are small and MDP is known we find optimal policy in poly-time via LP
  - Can also use value iteration or policy Iteration
- We are interested in the case of exponentially large state spaces.

Large Worlds: Model-Based Approach

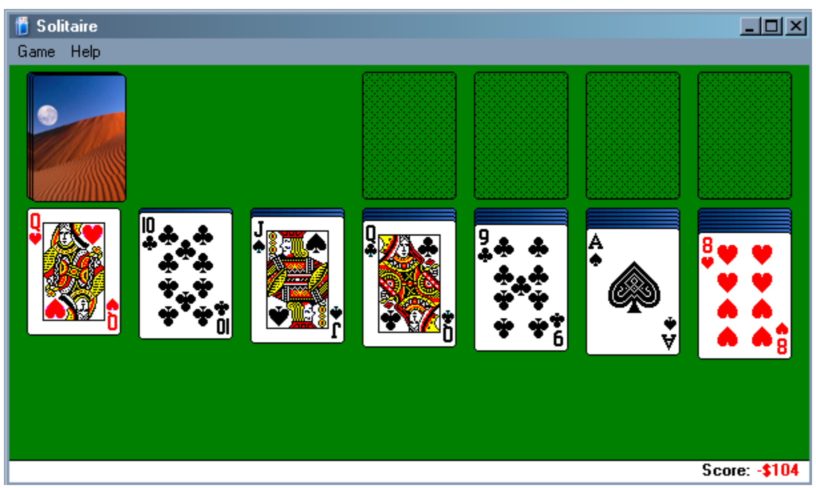
1. Define a language for compactly describing MDP model, for example:
   - Dynamic Bayesian Networks
   - Probabilistic STRIPS/PDDL
2. Design a planning algorithm for that language

Problem: more often than not, the selected language is inadequate for a particular problem, e.g.
- Problem size blows up
- Fundamental representational shortcoming

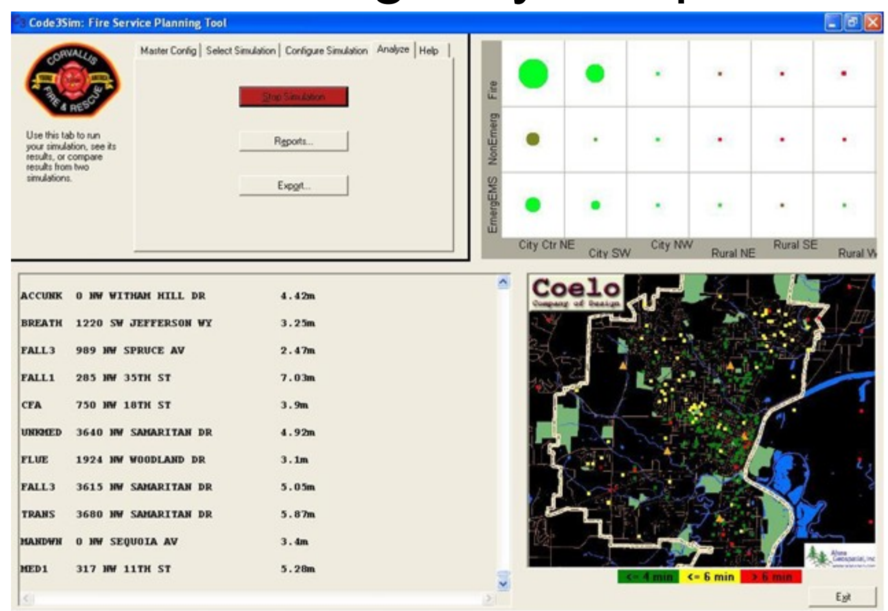
Large Worlds: Monte-Carlo Approach

- Often a simulator of a planning domain is available or can be learned from data
  - Even when domain can't be expressed via MDP language

Klondike Solitaire

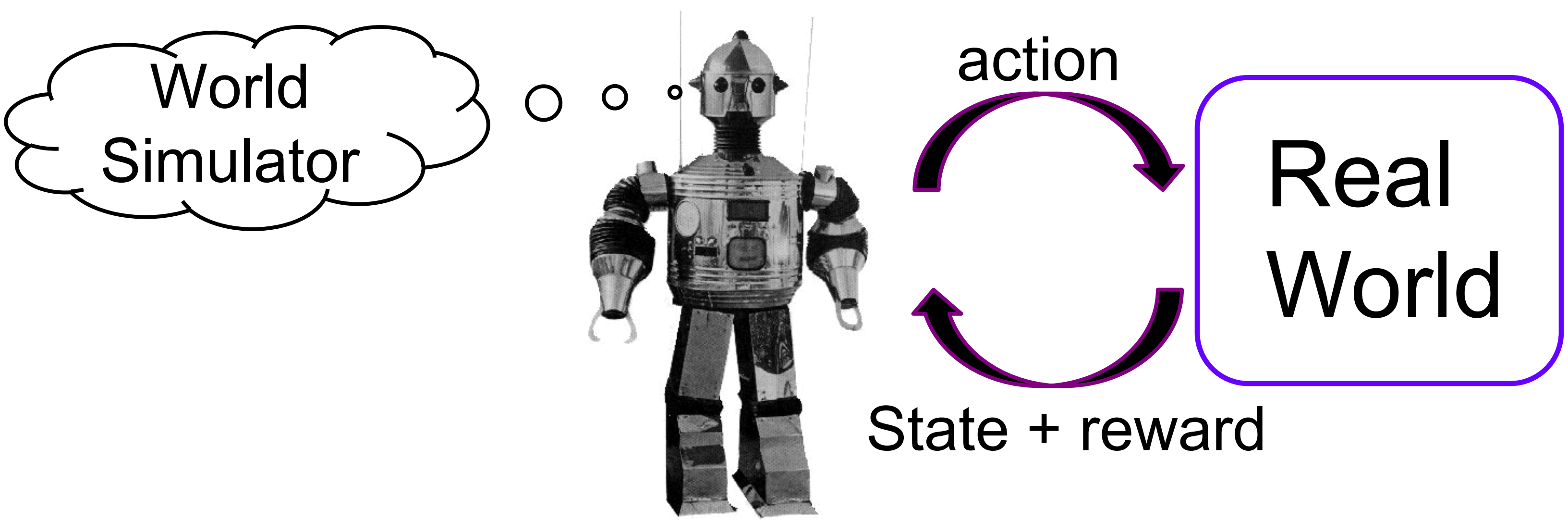


Fire & Emergency Response



Large Worlds: Monte-Carlo Approach

- Often a simulator of a planning domain is available or can be learned from data
  - Even when domain can't be expressed via MDP language
- **Monte-Carlo Planning**: compute a good policy for an MDP by interacting with an MDP simulator



Example Domains with Simulators

- Traffic simulators
- Robotics simulators
- Military campaign simulators
- Computer network simulators
- Emergency planning simulators
  - large-scale disaster and municipal
- Sports domains (Madden Football)
- Board games / Video games
  - Go / RTS

In many cases Monte-Carlo techniques yield state-of-the-art performance. Even in domains where model-based planner is applicable.

MDP: Simulation-Based Representation

- A simulation-based representation gives: S, A, R, T:
  - finite state set S (generally very large)
  - finite action set A
  - Stochastic, real-valued, bounded reward function R(s,a) = r
    - Stochastically returns a reward **r** given input s and a
    - Can be implemented in arbitrary programming language
  - Stochastic transition function T(s,a) = s' (i.e. a simulator)
    - Stochastically returns a state **s'** given input s and a
    - Probability of returning **s'** is dictated by Pr(s' | s,a) of MDP
    - T can be implemented in an arbitrary programming language

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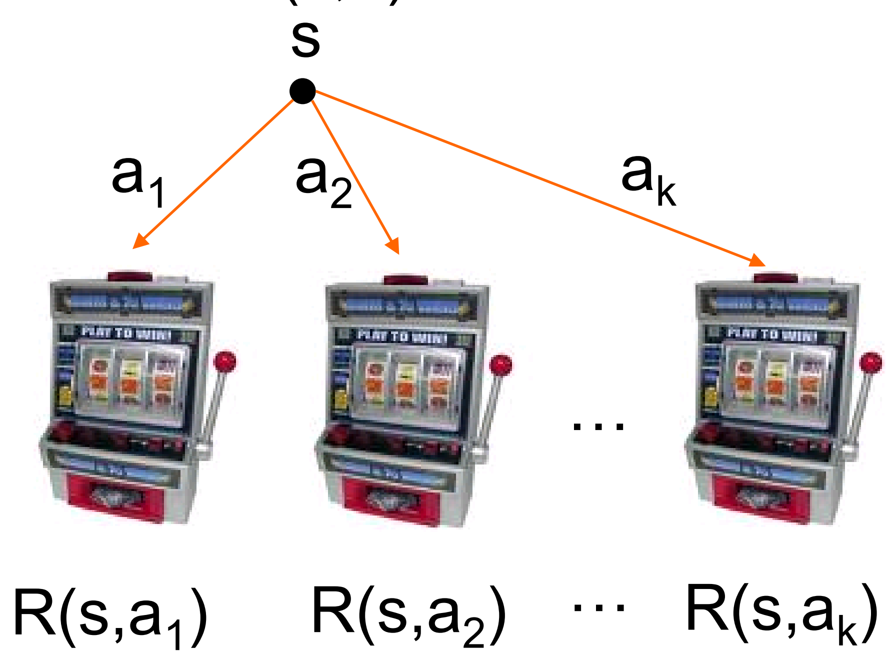
Single State Monte-Carlo Planning

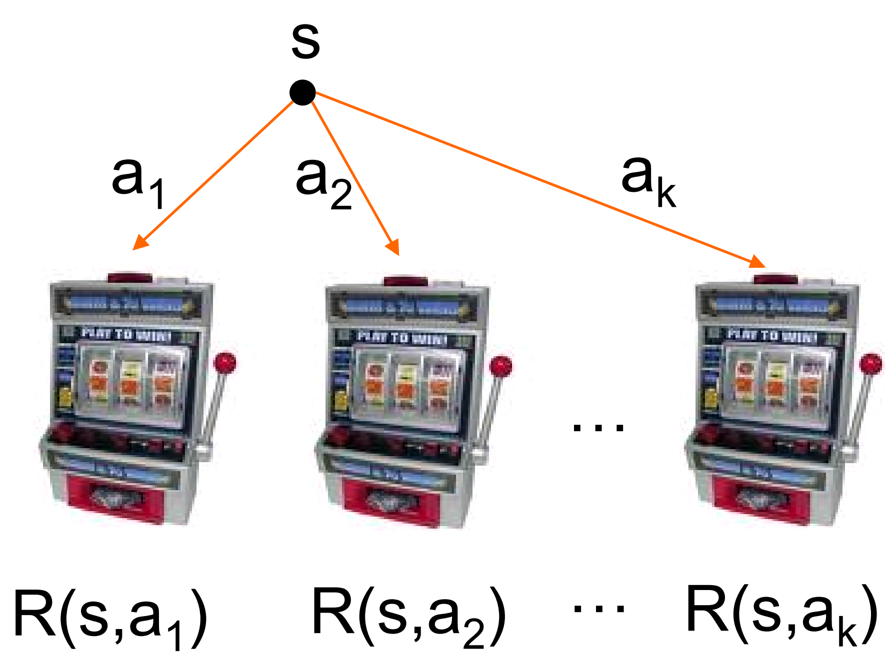
- Suppose MDP has a single state and k actions
  - Figure out which action has best expected reward
  - Can sample rewards of actions using calls to simulator
  - Sampling a is like pulling slot machine arm with random payoff function R(s,a)



Multi-Armed Bandit Problem

PAC Bandit Objective

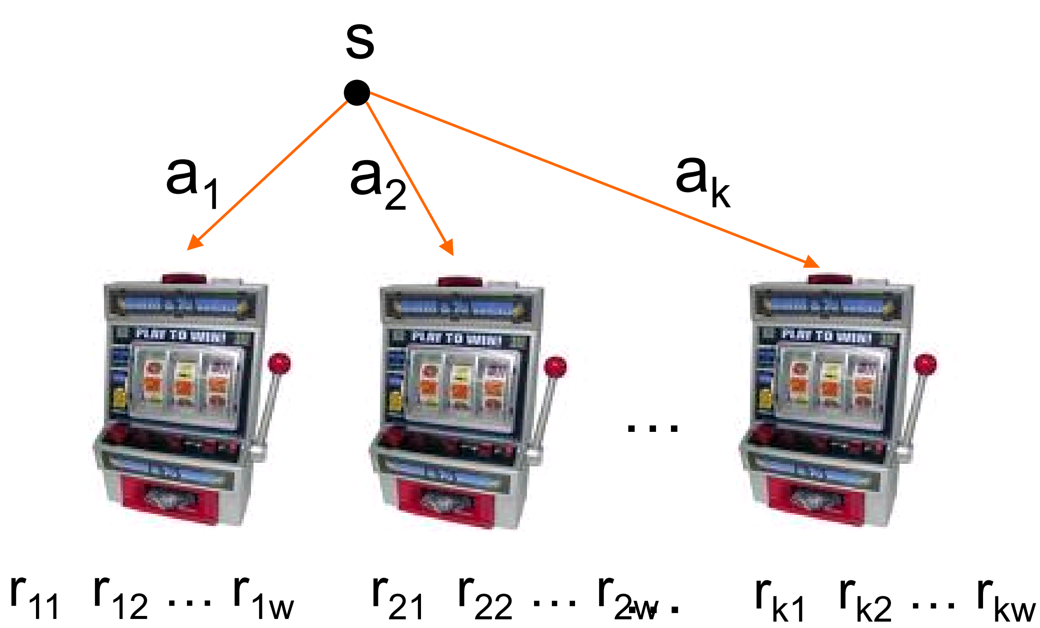
- **Probably Approximately Correct (PAC)**
  - Select an arm that probably (w/ high probability) has approximately the best expected reward
  - Use as few simulator calls (or pulls) as possible



Multi-Armed Bandit Problem

UniformBandit Algorithm
NaiveBandit from [Even-Dar et. al., 2002]

1. Pull each arm **w** times (uniform pulling).
2. Return arm with best average reward.



How large must w be to provide a PAC guarantee?

Aside: Additive Chernoff Bound

- Let R be a random variable with maximum absolute value Z. An let r_i i=1,…,w be i.i.d. samples of R
- The Chernoff bound gives a bound on the probability that the average of the r_i are far from E[R]

Chernoff Bound

$$\Pr\left(\left|E[R]-\tfrac{1}{w}\sum_{i=1}^{w} r_i\right|\geq \varepsilon\right)\leq \exp\left(-\left(\frac{\varepsilon}{Z}\right)^2 w\right)$$

Equivalently:

With probability at least $1-\delta$ we have that,

$$\left|E[R]-\tfrac{1}{w}\sum_{i=1}^{w} r_i\right|\leq Z\sqrt{\tfrac{1}{w}\ln\tfrac{1}{\delta}}$$

UniformBandit Algorithm

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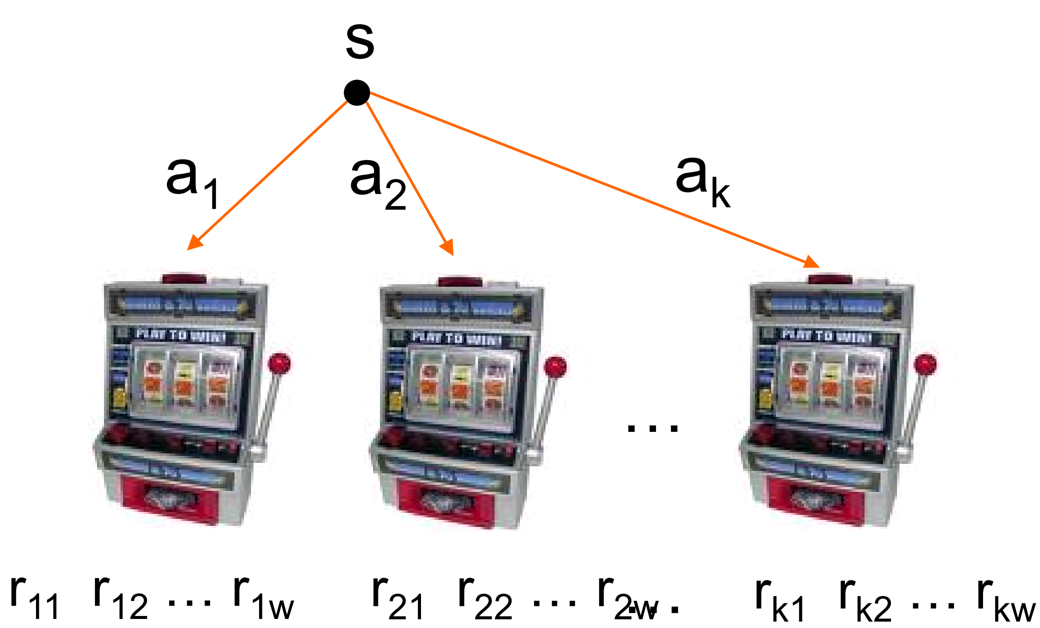


How large must w be to provide a PAC guarantee?

UniformBandit PAC Bound

With a bit of algebra and Chernoff bound we get:

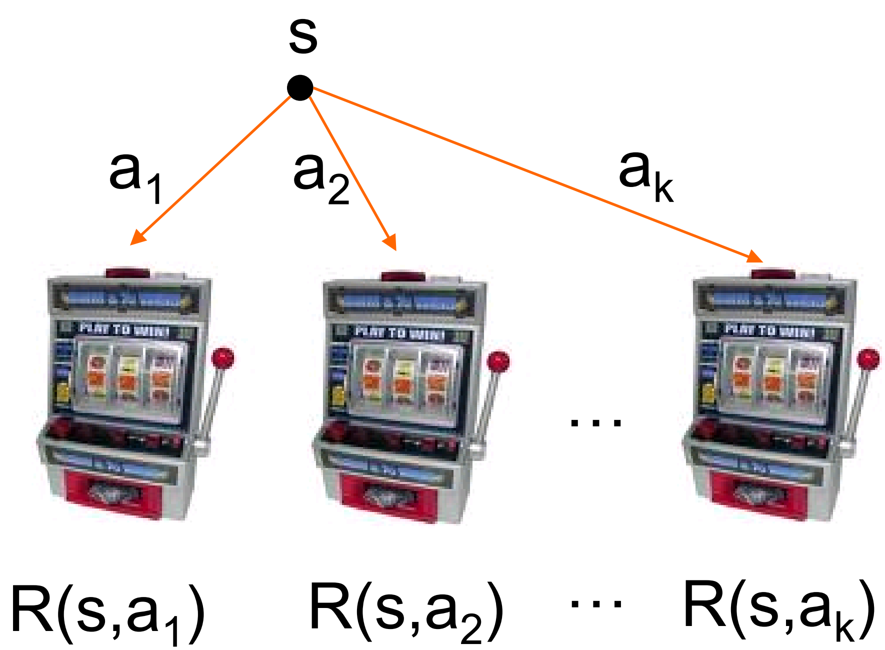
If $w \geq \left(\frac{R_{\max}}{\varepsilon}\right)^2 \ln \frac{k}{\delta}$ for all arms simultaneously

$$\left| E[R(s,a_i)] - \frac{1}{w}\sum_{j=1}^{w} r_{ij} \right| \leq \varepsilon$$

with probability at least $1-\delta$

- That is, estimates of all actions are ε – accurate with probability at least 1- δ
- Thus selecting estimate with highest value is approximately optimal with high probability, or PAC

Simulator Calls for UniformBandit



- Total simulator calls for PAC: $k \cdot w = O\left(\frac{k}{\varepsilon^2} \ln \frac{k}{\delta}\right)$

- Can get rid of ln(k) term with more complex algorithm [Even-Dar et. al., 2002].

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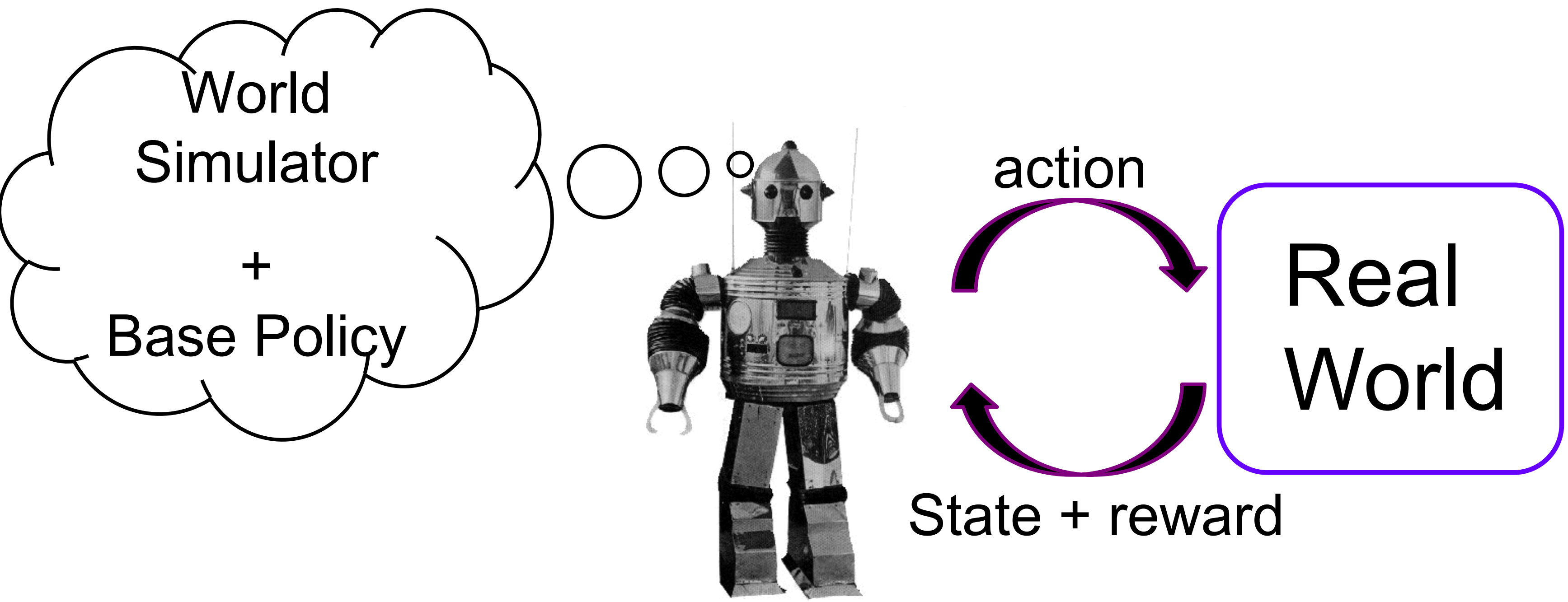
Policy Improvement via Monte-Carlo

- Now consider a multi-state MDP.
- Suppose we have a simulator and a non-optimal policy
  - E.g. policy could be a standard heuristic or based on intuition
- Can we somehow compute an improved policy?



Policy Improvement Theorem

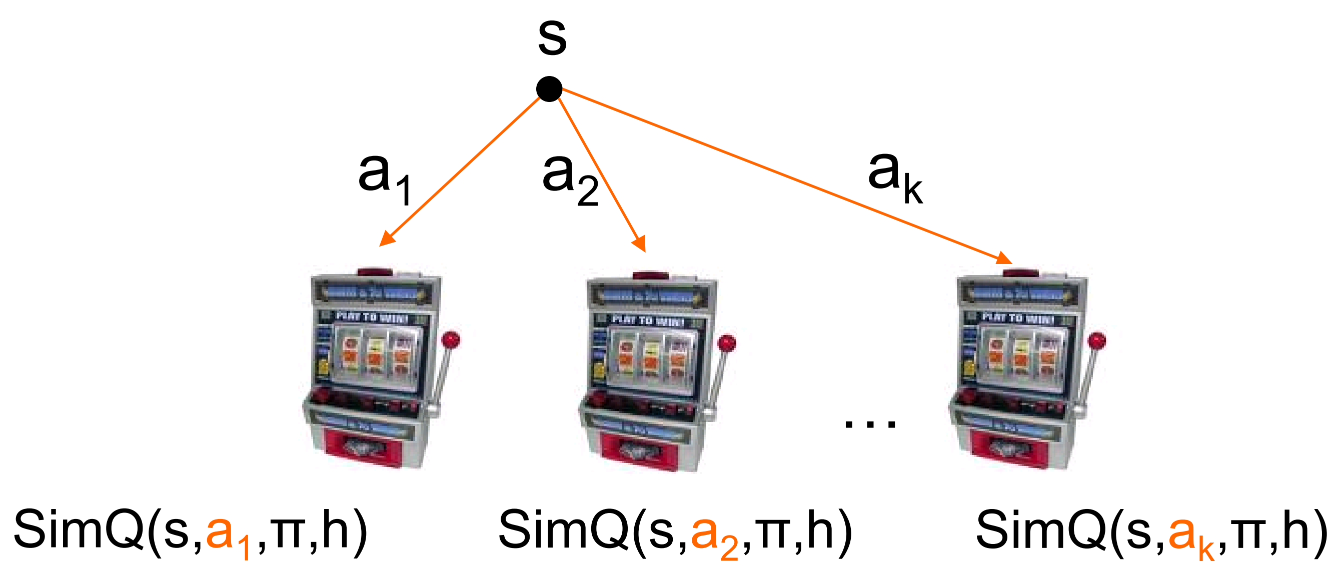
- The h-horizon Q-function $Q_\pi(s,a,h)$ is defined as:
  expected reward of starting in state s, taking action a, and then following policy π for h-1 steps

- **Define:** $\pi'(s) = \arg\max_a Q_\pi(s,a,h)$

- **Theorem [Howard, 1960]:** For any non-optimal policy π the policy π' a strict improvement over π.

- Computing π' amounts to finding the action that maximizes the Q-function
  - Can we use the bandit idea to solve this?

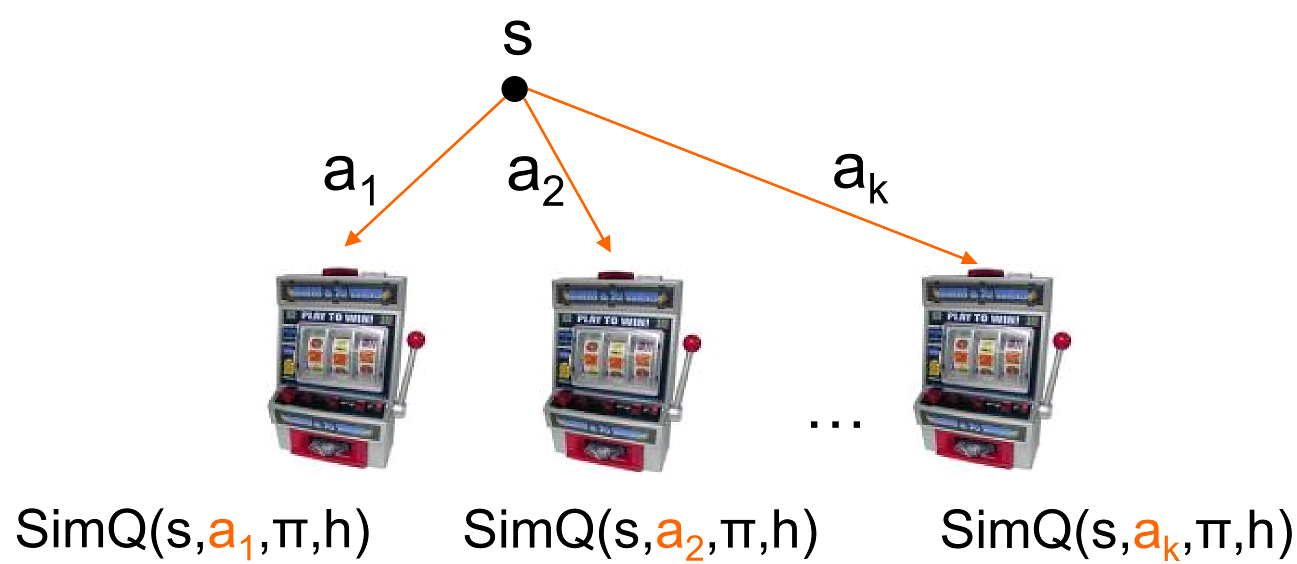
Policy Improvement via Bandits



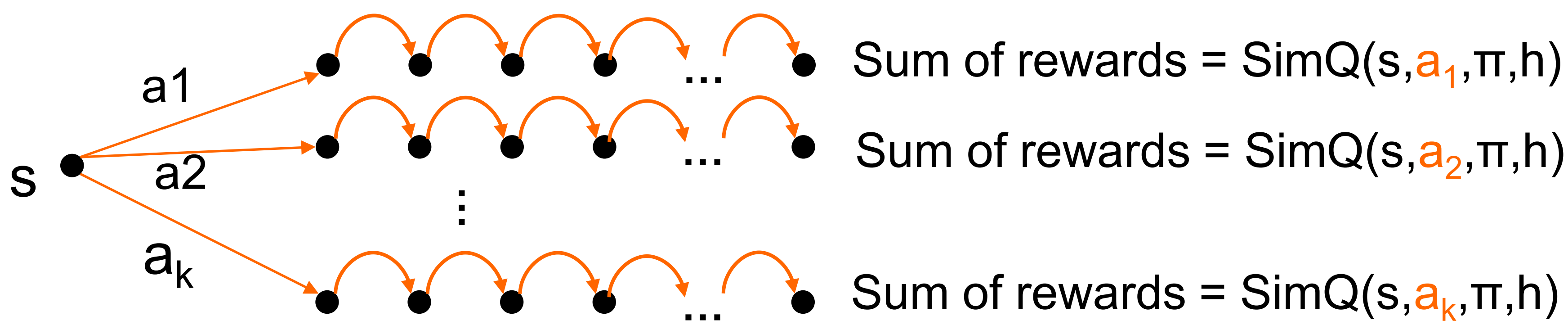
- **Idea:** define a stochastic function **SimQ(s,a,π,h)** whose expected value is $Q_\pi(s,a,h)$
- Use Bandit algorithm to PAC select improved action

How to implement SimQ?

Policy Improvement via Bandits

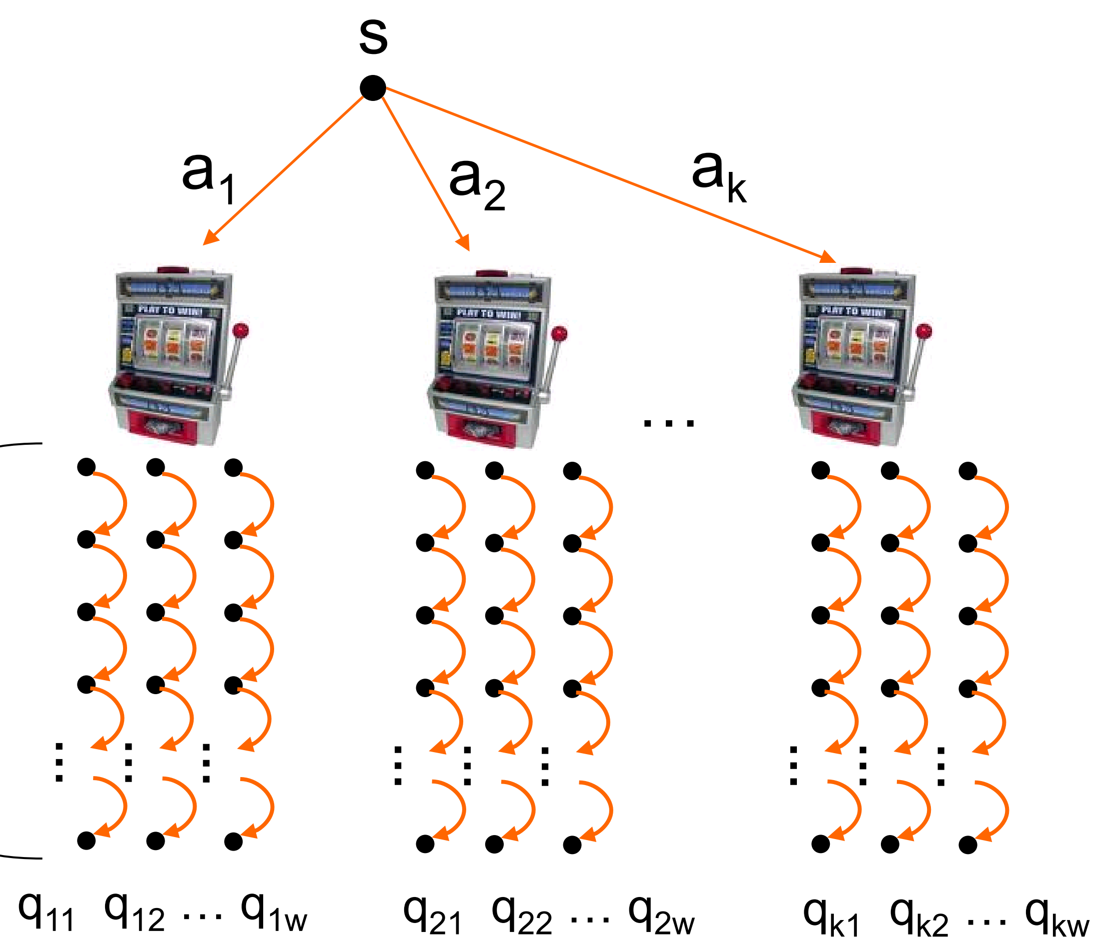


Trajectory under π



Policy Rollout Algorithm

1. For each a_i run SimQ(s,a_i,π,h) **w** times
2. Return action with best average of SimQ results



SimQ(s,a_i,π,h) trajectories

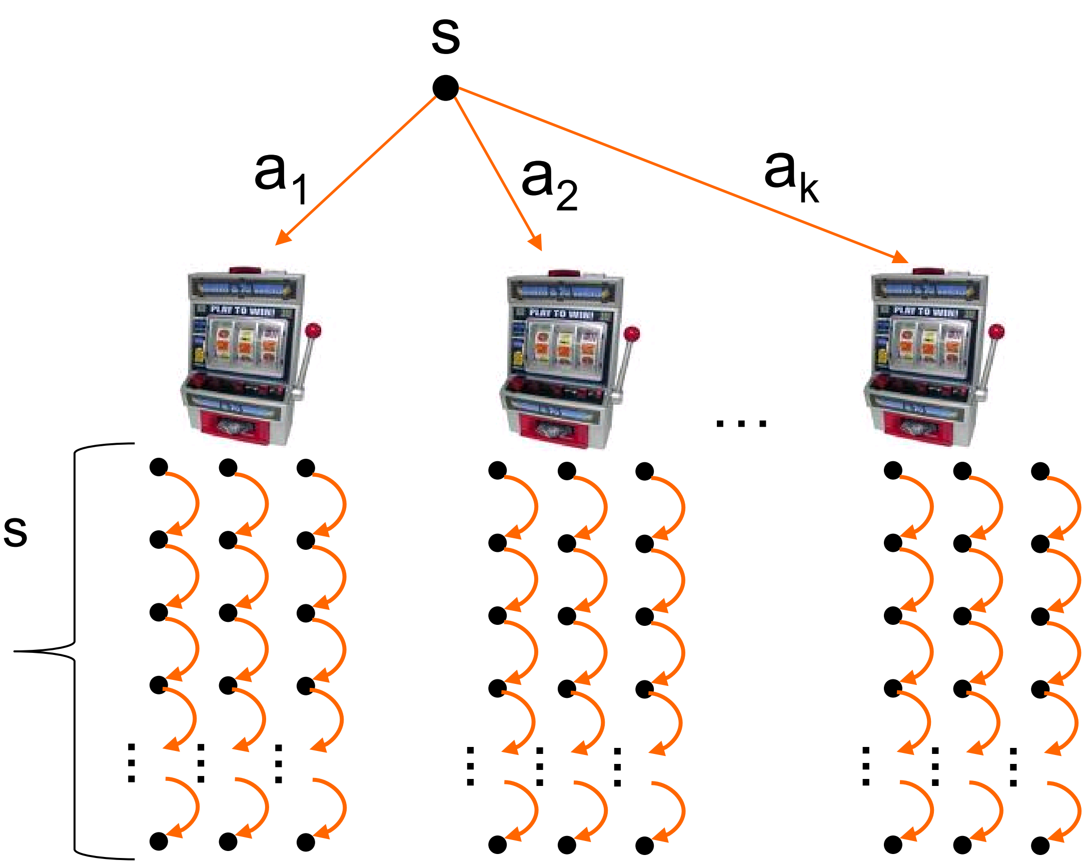
Each simulates taking action a_i then following π for h-1 steps.

Samples of SimQ(s,a_i,π,h)

Policy Rollout: # of Simulator Calls

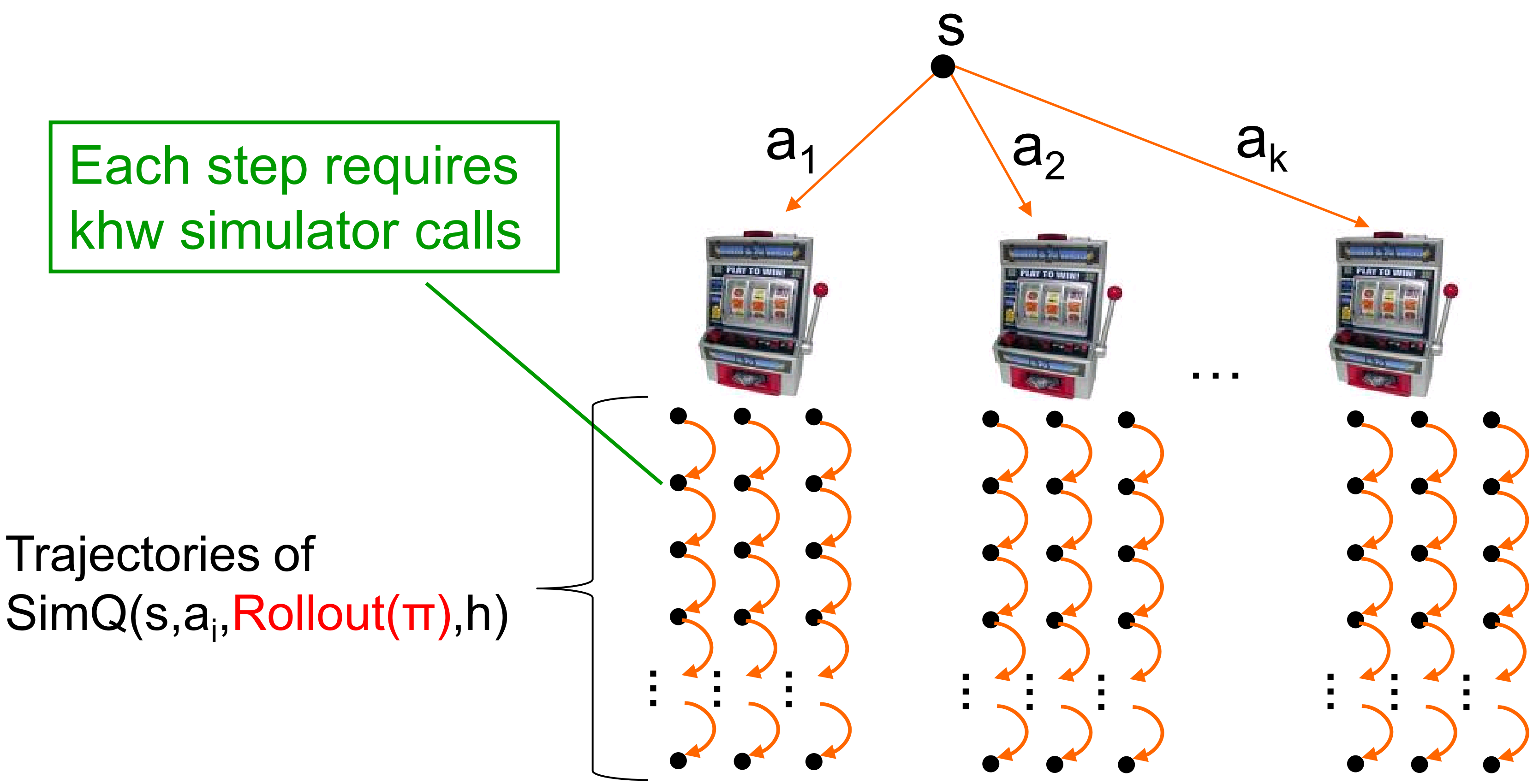
SimQ(s,a_i,π,h) trajectories

Each simulates taking action a_i then following π for h-1 steps.



- For each action **w** calls to SimQ, each using h sim calls
- Total of khw calls to the simulator

Multi-Stage Rollout



- Two stage: compute rollout policy of rollout policy of π
- Requires $(khw)^2$ calls to the simulator for 2 stages
- In general exponential in the number of stages

Rollout Summary

- We often are able to write simple, mediocre policies
  - Network routing policy
  - Policy for card game of Hearts
  - Policy for game of Backgammon
  - Solitaire playing policy
- Policy rollout is a general and easy way to improve upon such policies
- Often observe substantial improvement, e.g.
  - Compiler instruction scheduling
  - Backgammon
  - Network routing
  - Combinatorial optimization
  - Game of GO
  - Solitaire

Example: Rollout for Thoughtful Solitaire
[Yan et al. NIPS'04]

Player	Success Rate	Time/Game
Human Expert	36.6%	20 min
(naïve) Base Policy	13.05%	0.021 sec
1 rollout	31.20%	0.67 sec
2 rollout	47.6%	7.13 sec
3 rollout	56.83%	1.5 min
4 rollout	60.51%	18 min
5 rollout	70.20%	1 hour 45 min

- Multiple levels of rollout can payoff but is expensive

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Sparse Sampling

- Rollout does not guarantee optimality or near optimality

- Can we develop simulation-based methods that give us near optimal policies?
  - With computation that doesn't depend on number of states!

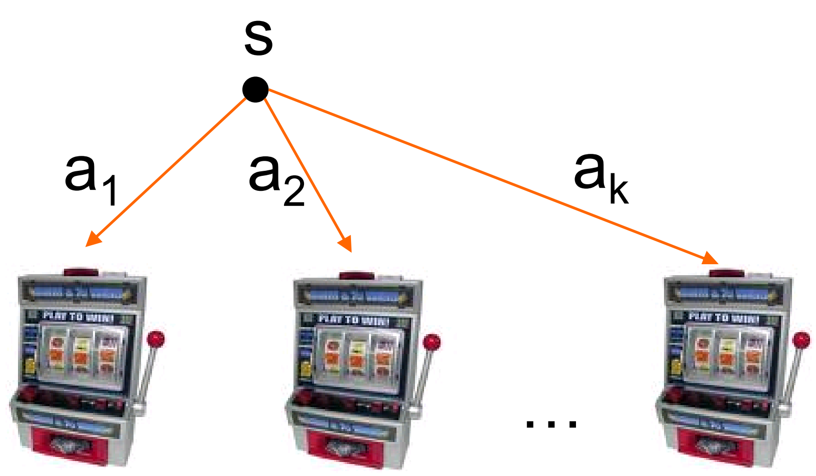
- In deterministic games and problems it is common to build a look-ahead tree at a state to determine best action
  - Can we generalize this to general MDPs?

- **Sparse Sampling** is one such algorithm
  - Strong theoretical guarantees of near optimality

MDP Basics

- Let $V^*(s,h)$ be the optimal value function of MDP
- Define $Q^*(s,a,h) = E[R(s,a) + V^*(T(s,a),h-1)]$
  - Optimal h-horizon value of action *a* at state *s.*
  - R(s,a) and T(s,a) return random reward and next state
- **Optimal Policy:** $\pi^*(x) = \mathrm{argmax}_a\, Q^*(x,a,h)$
- What if we knew V^*?
  - Can apply bandit algorithm to select action that approximately maximizes $Q^*(s,a,h)$

Bandit Approach Assuming V*



Return sample of
$R(s, a_i) + V^*(T(s, a_i), h-1)$

$SimQ^*(s,a_1,h)$ $SimQ^*(s,a_2,h)$ $SimQ^*(s,a_k,h)$

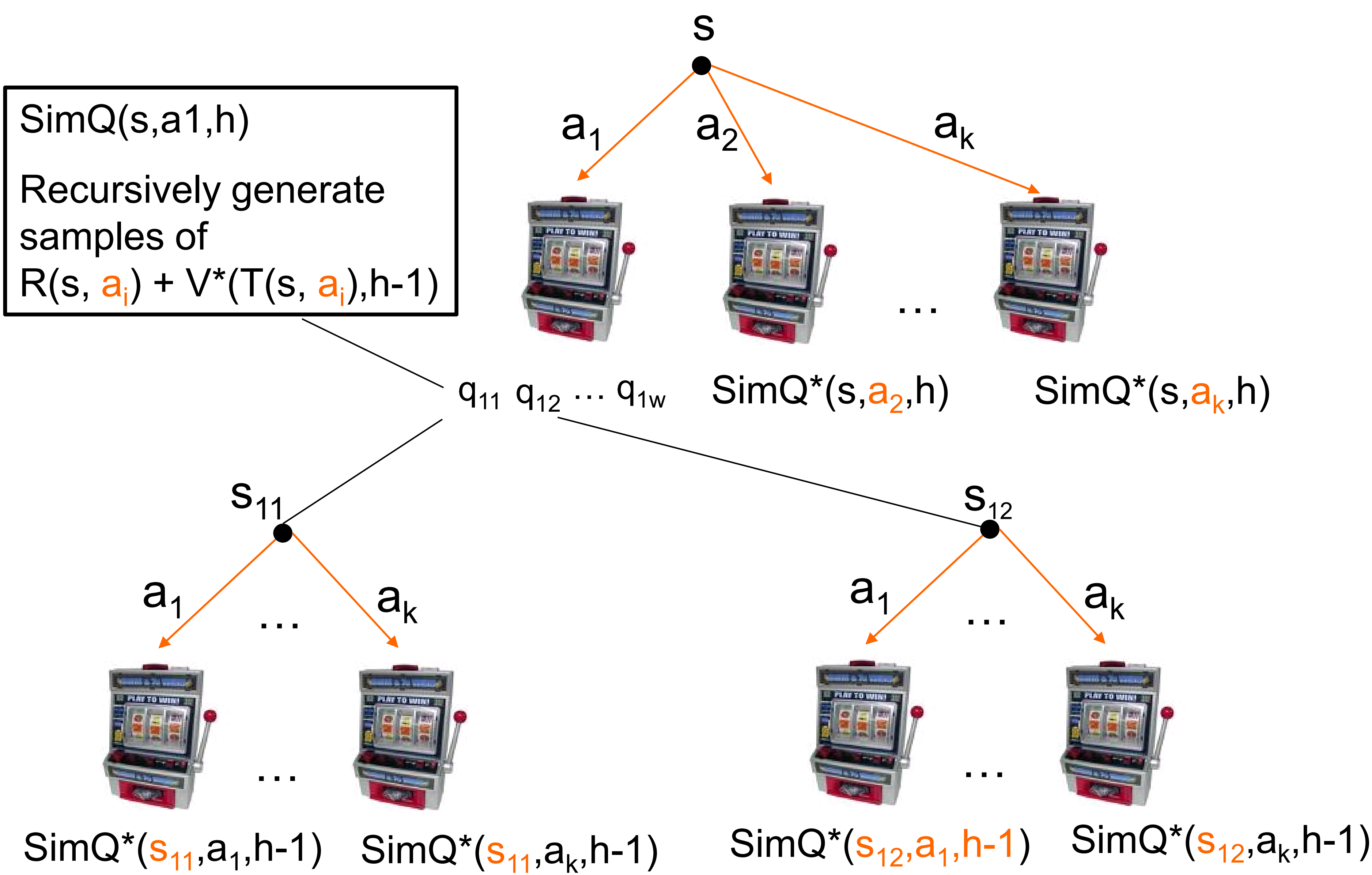
SimQ*(s,a,h)
  s' = T(s,a)
  r = R(s,a)
  Return r + V*(s',h-1)

- Expected value of SimQ*(s,a,h) is Q*(s,a,h)
  - Use UniformBandit to select approximately optimal action

But we don't know V*

- To compute SimQ*(s,a,h) need V*(s',h-1) for any s'
- Use recursive identity (Bellman's equation):
  - $V^*(s,h\text{-}1) = \max_a Q^*(s,a,h\text{-}1)$
- **Idea:** Can recursively estimate V*(s,h-1) by running h-1 horizon bandit based on SimQ*
- **Base Case:** V*(s,0) = 0, for all s

Recursive UniformBandit



Sparse Sampling [Kearns et. al. 2002]

This recursive UniformBandit is called Sparse Sampling

Return value estimate V*(s,h) of state s and estimated optimal action a*

SparseSampleTree(*s*,*h*,*w*)

For each action *a* in *s*

    Q*(s,a,h) = 0

    For i = 1 to w

        Simulate taking *a* in *s* resulting in s_i and reward r_i

        [V*(s_i,h),a*] = **SparseSample**(s_i,h-1,w)

        Q*(s,a,h) = Q*(s,a,h) + r_i + V*(s_i,h)

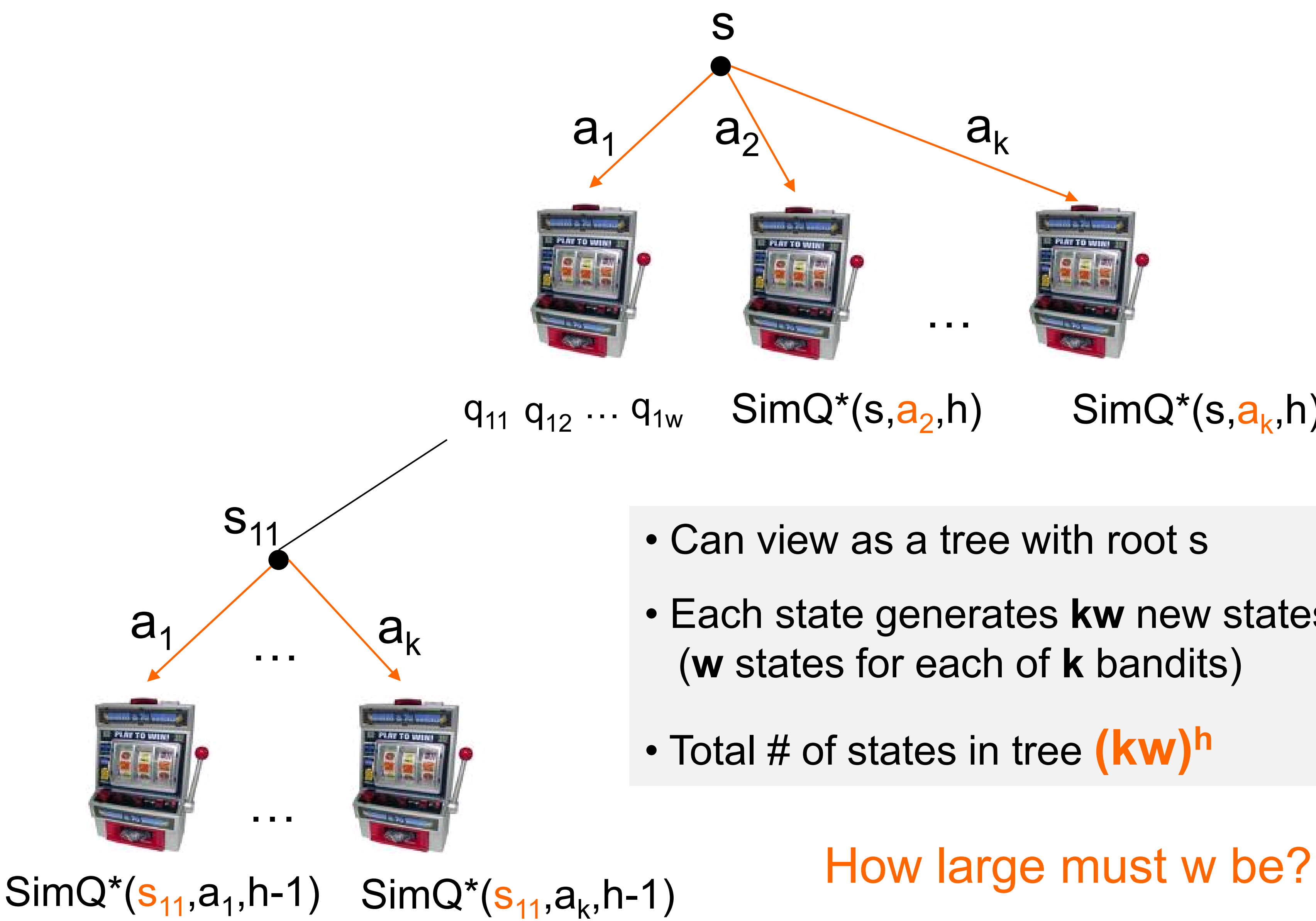
    Q*(s,a,h) = Q*(s,a,h) / w   ;; estimate of Q*(s,a,h)

V*(s,h) = $\max_a$ Q*(s,a,h)   ;; estimate of V*(s,h)

a* = argmax_a Q*(s,a,h)

Return [V*(s,h), a*]

of Simulator Calls



- Can view as a tree with root s
- Each state generates **kw** new states (**w** states for each of **k** bandits)
- Total # of states in tree $(\mathbf{kw})^{\mathbf{h}}$

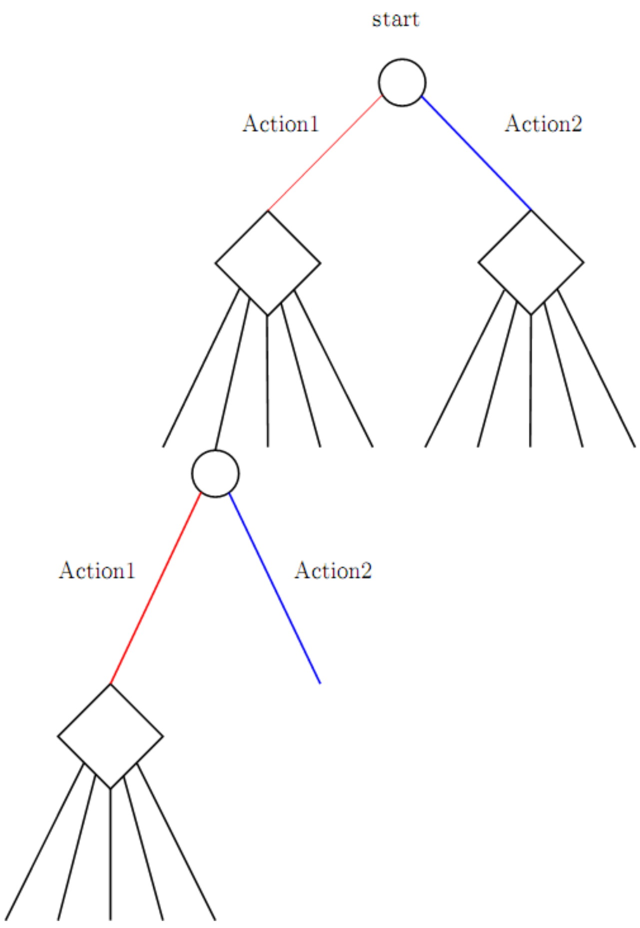
How large must w be?

Sparse Sampling

- For a given desired accuracy, how large should sampling width and depth be?
  - Answered: [Kearns et. al., 2002]
- **Good news:** can achieve near optimality for value of w independent of state-space size!
  - First near-optimal general MDP planning algorithm whose runtime didn't depend on size of state-space
- **Bad news:** the theoretical values are typically still intractably large---also exponential in **h**
- **In practice:** use small **h** and use heuristic at leaves (similar to minimax game-tree search)

Uniform vs. Adaptive Bandits

- Sparse sampling wastes time on bad parts of tree
  - Devotes equal resources to each state encountered in the tree
  - Would like to focus on most promising parts of tree
- But how to control exploration of new parts of tree vs. exploiting promising parts?
- Need adaptive bandit algorithm that explores more effectively



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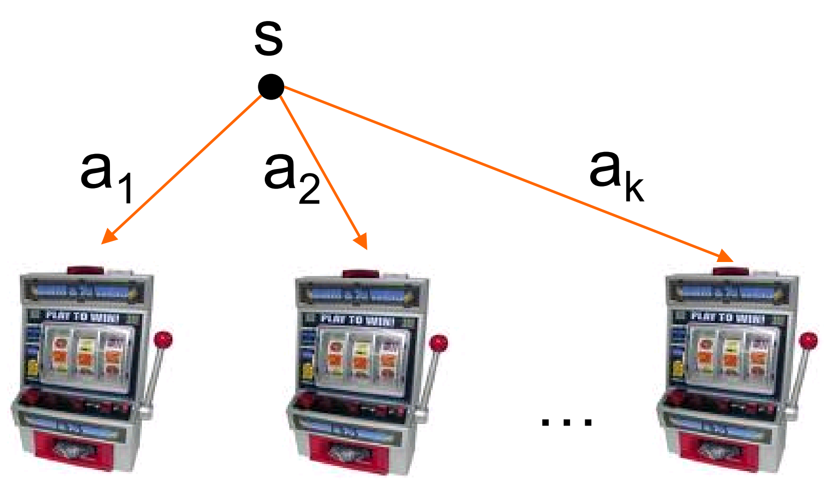
Regret Minimization Bandit Objective

- **Problem:** find arm-pulling strategy such that the expected total reward at time n is close to the best possible (i.e. pulling the best arm always)
  - UniformBandit is poor choice --- waste time on bad arms
  - Must balance **exploring** machines to find good payoffs and **exploiting** current knowledge



UCB Adaptive Bandit Algorithm
[Auer, Cesa-Bianchi, & Fischer, 2002]

- Q(a) : average payoff for action a based on current experience
- n(a) : number of pulls of arm a
- Action choice by UCB after n pulls:

Assumes payoffs in [0,1]

$$a^* = \arg\max_a Q(a) + \sqrt{\frac{2 \ln n}{n(a)}}$$

- **Theorem**: The expected regret after *n* arm pulls compared to optimal behavior is bounded by O(log *n*)
- No algorithm can achieve a better loss rate

UCB Algorithm [Auer, Cesa-Bianchi, & Fischer, 2002]

$$a^* = \arg\max_a Q(a) + \sqrt{\frac{2\ln n}{n(a)}}$$

Value Term:
favors actions that looked good historically

Exploration Term:
actions get an exploration bonus that grows with ln(n)

Expected number of pulls of sub-optimal arm **a** is bounded by:

$$\frac{8}{\Delta_a^2}\ln n$$

where Δ_a is regret of arm **a**

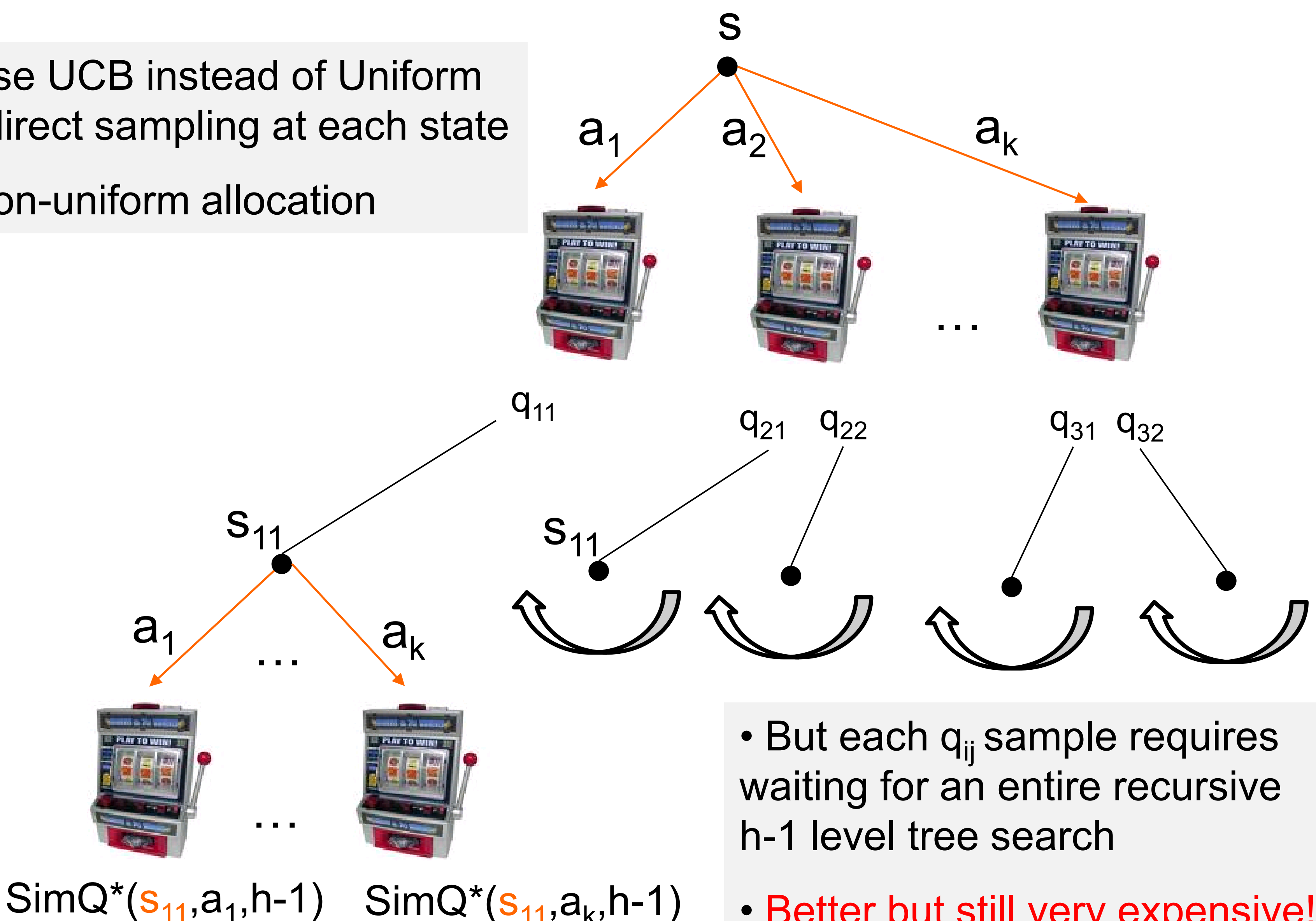
Doesn't waste much time on sub-optimal arms unlike uniform!

UCB for Multi-State MDPs

- UCB-Based Policy Rollout:
  - Use UCB to select actions instead of uniform
- UCB-Based Sparse Sampling
  - Use UCB to make sampling decisions at internal tree nodes

UCB-based Sparse Sampling [Chang et. al. 2005]

- Use UCB instead of Uniform to direct sampling at each state
- Non-uniform allocation



- But each q_{ij} sample requires waiting for an entire recursive h-1 level tree search
- Better but still very expensive!

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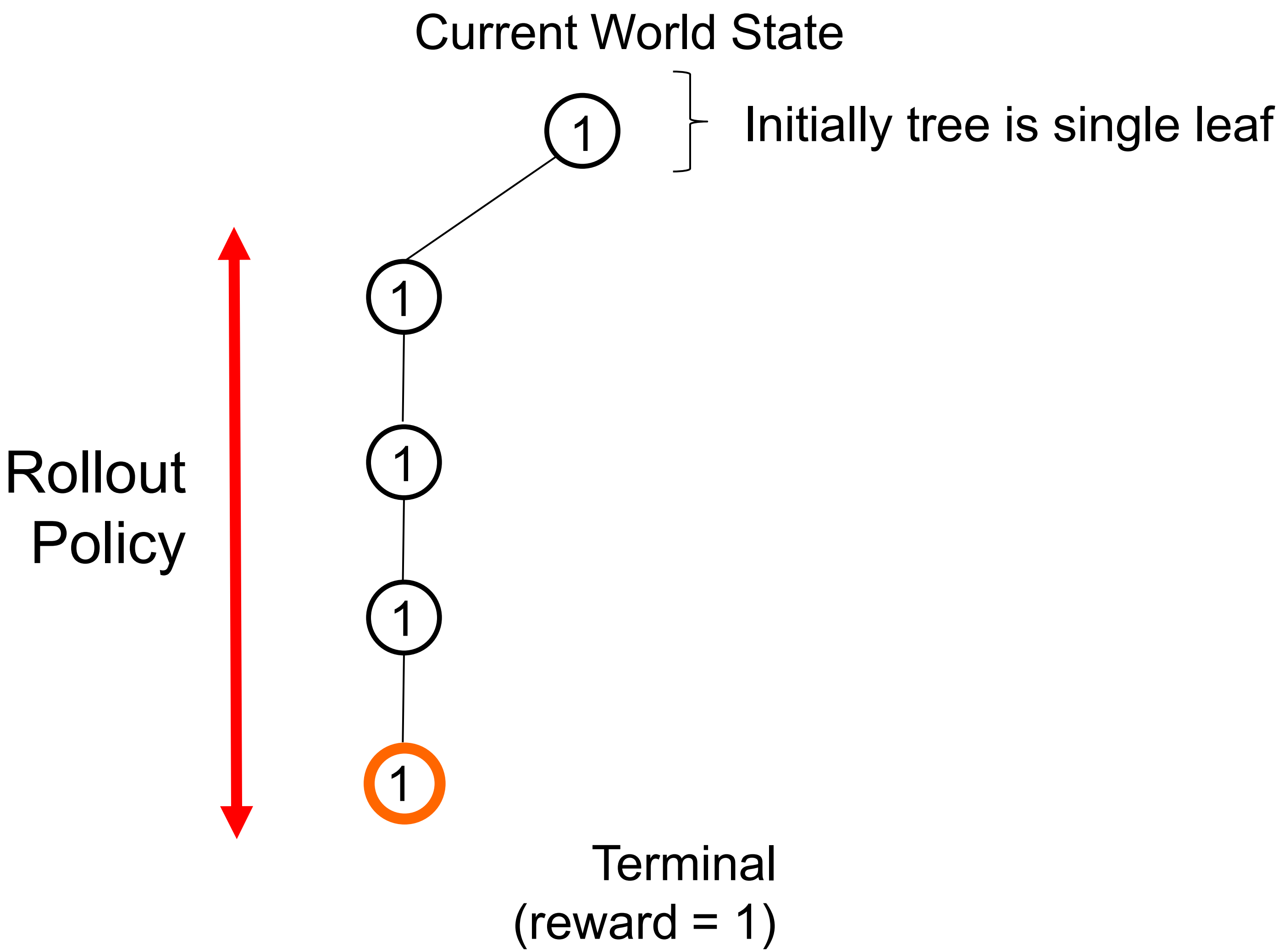
UCT Algorithm [Kocsis & Szepesvari, 2006]

- Instance of Monte-Carlo Tree Search
  - Applies principle of UCB
  - Some nice theoretical properties
  - Much better anytime behavior than sparse sampling
  - Major advance in computer Go

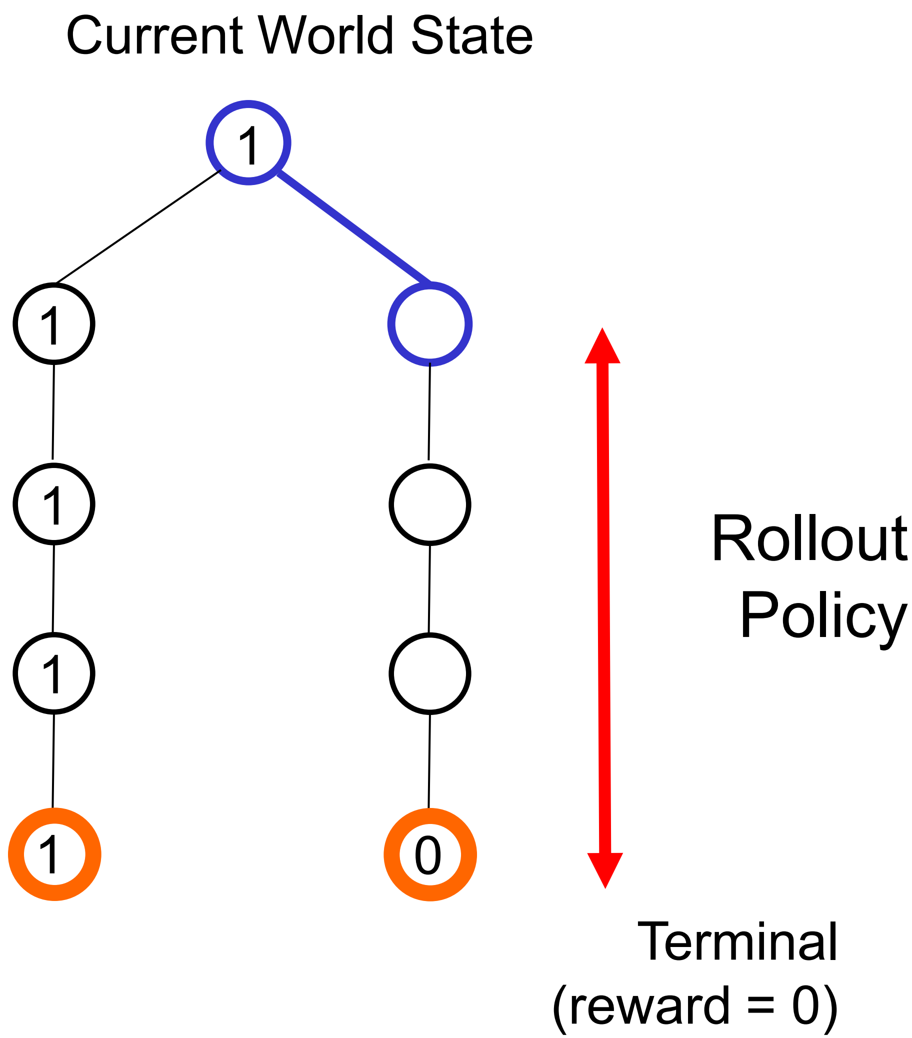
- Monte-Carlo Tree Search
  - Repeated Monte Carlo simulation of a rollout policy
  - Each rollout adds one or more nodes to search tree

- Rollout policy depends on nodes already in tree

At a leaf node perform a random rollout



Must select each action at a node at least once



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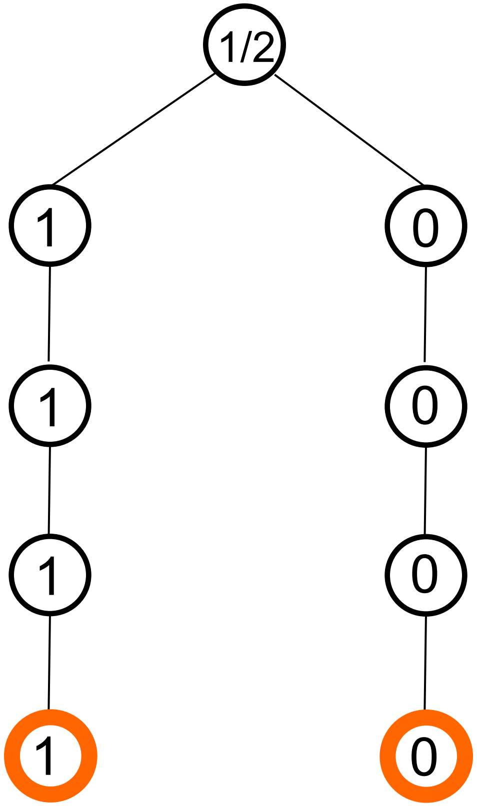
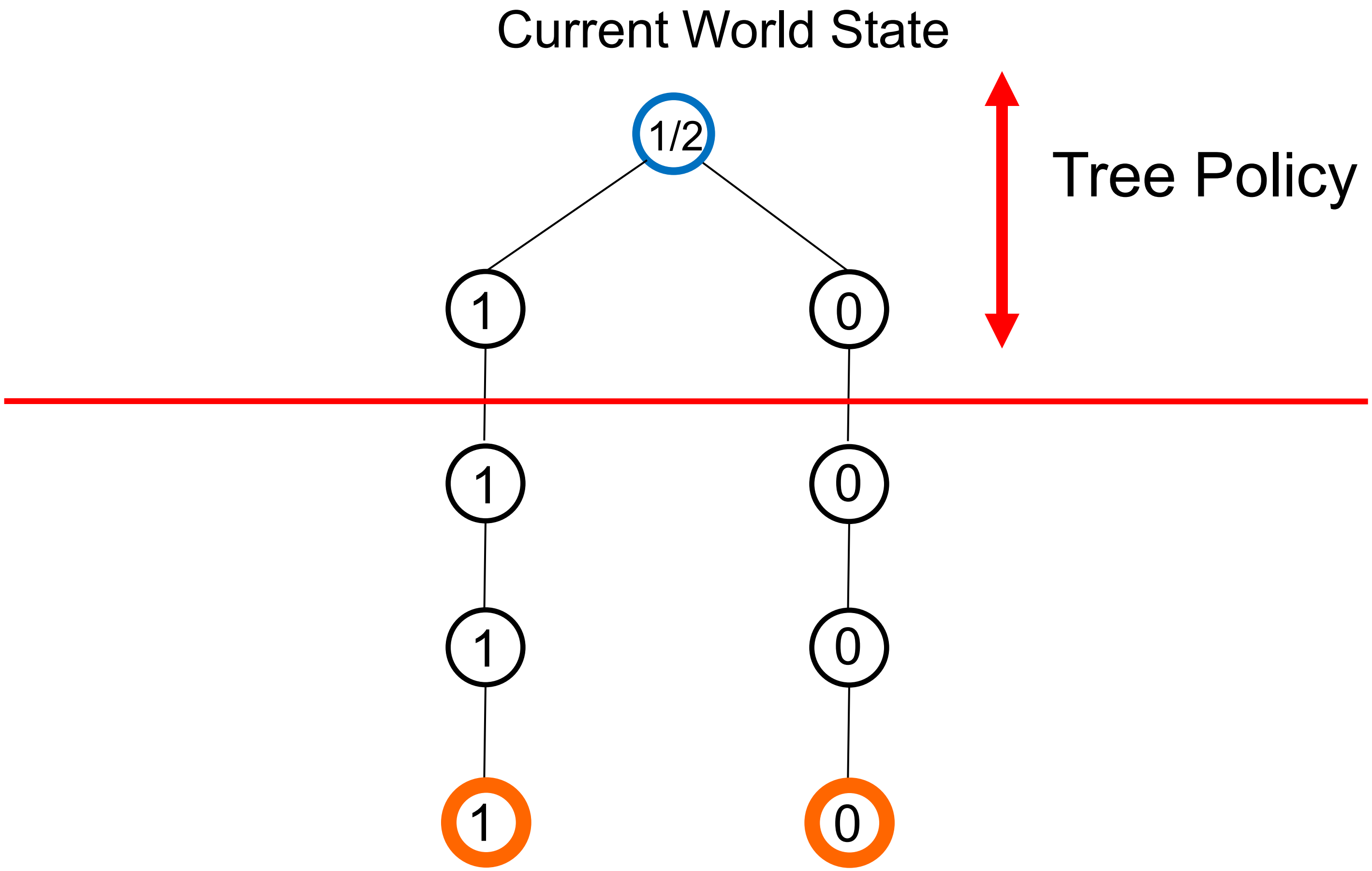
Current World State



When all node actions tried once, select action according to tree policy



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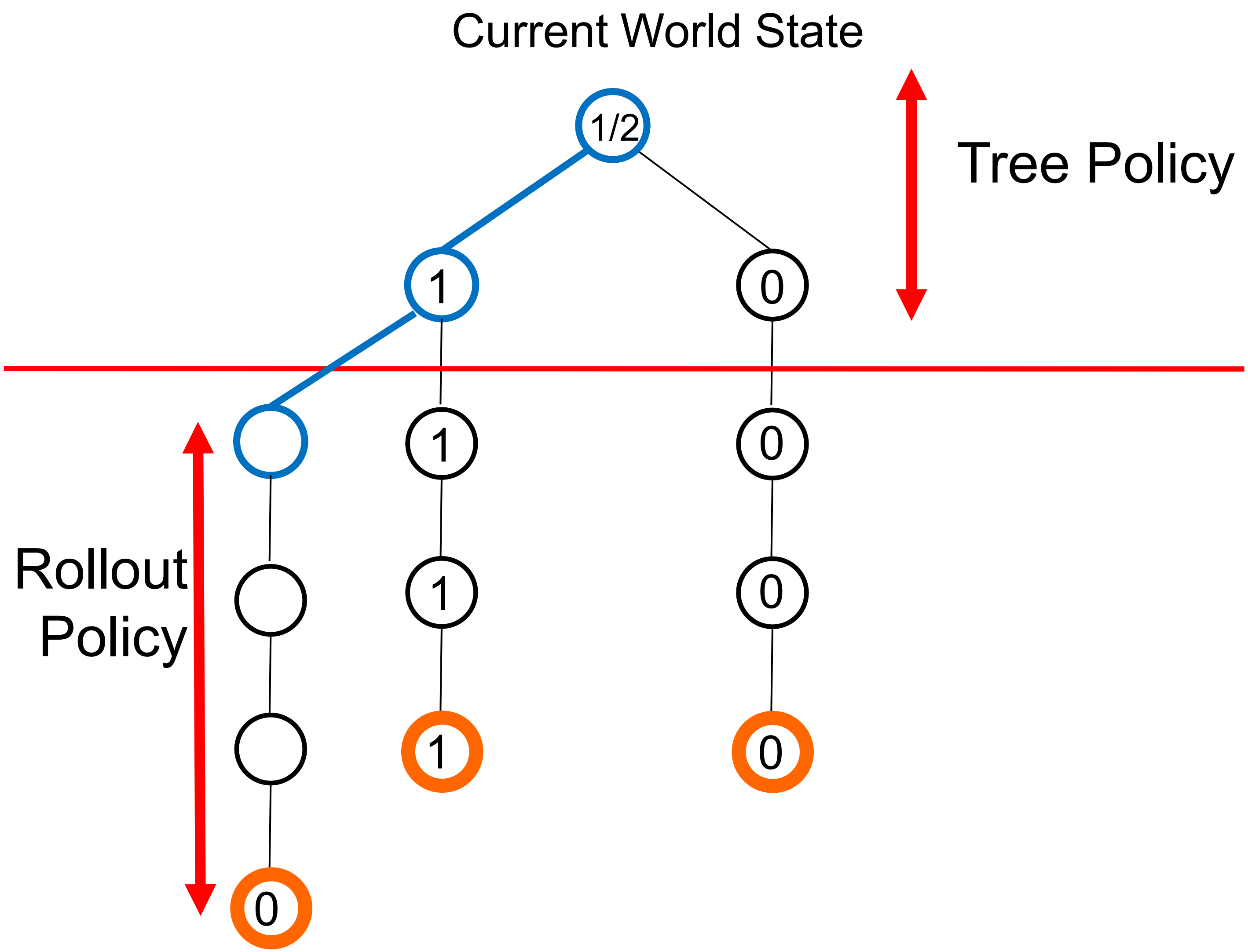
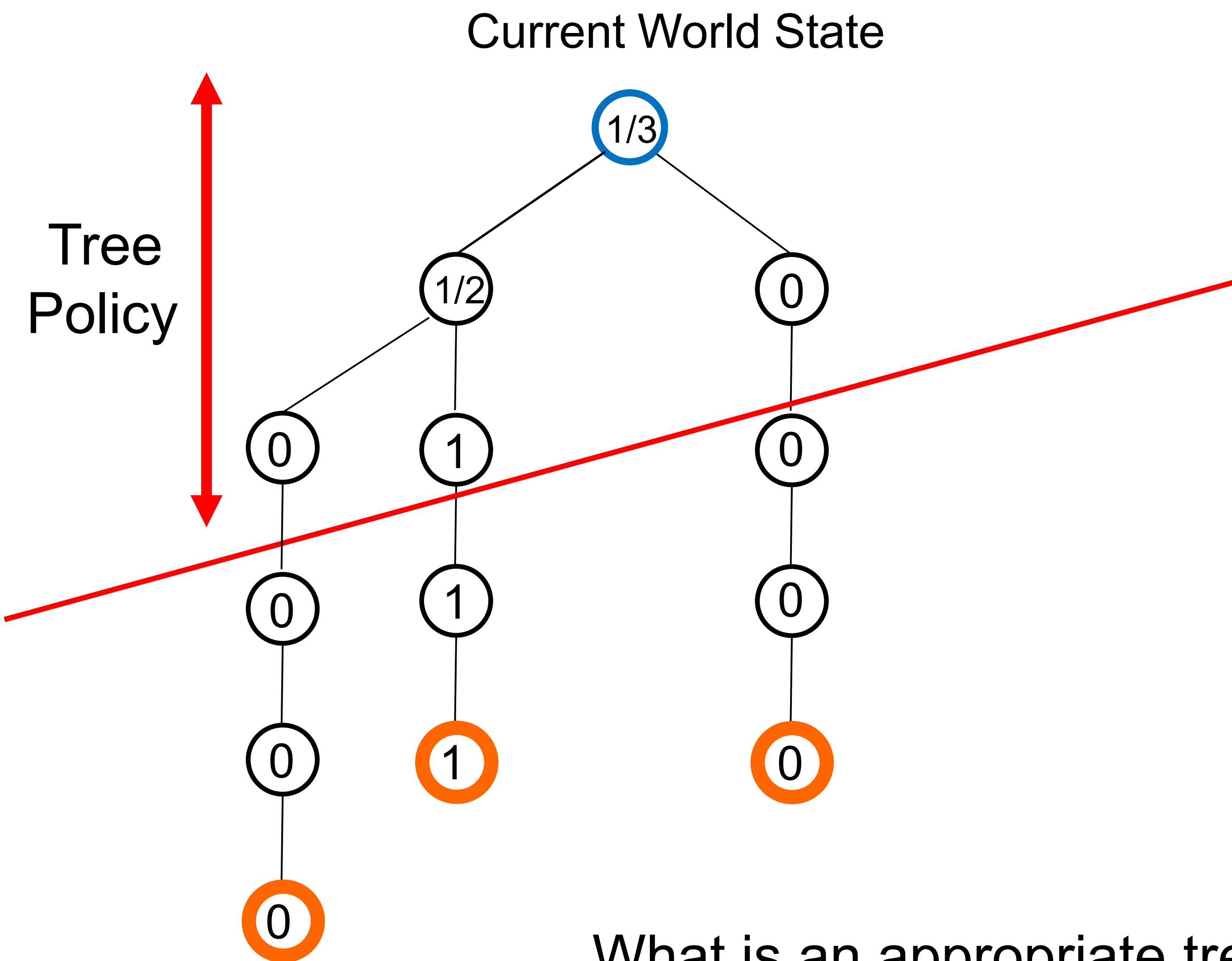


When all node actions tried once, select action according to tree policy



What is an appropriate tree policy?
Rollout policy?

UCT Algorithm [Kocsis & Szepesvari, 2006]

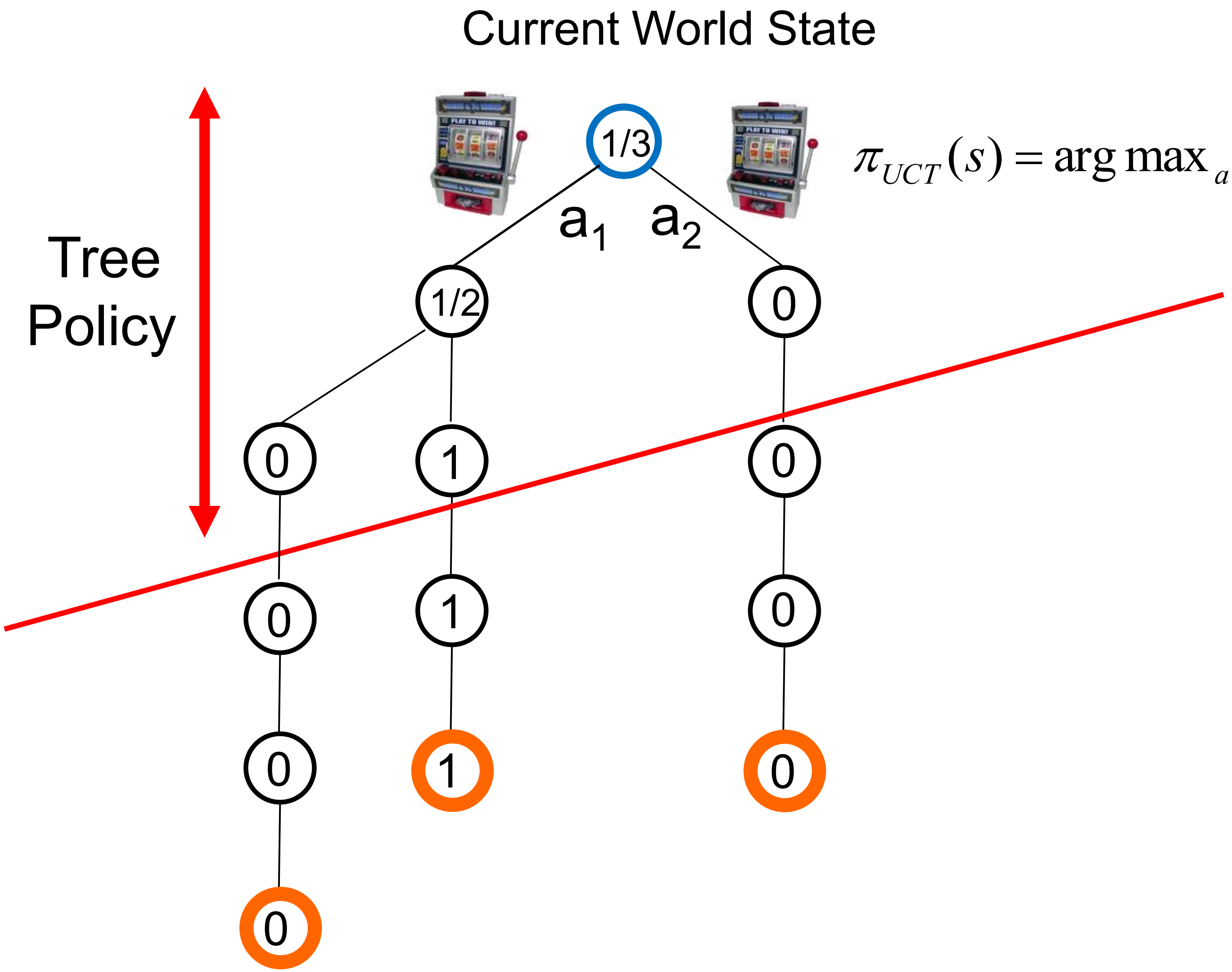
- Basic UCT uses random rollout policy

- Tree policy is based on UCB:
  - Q(s,a) : average reward received in current trajectories after taking action a in state s
  - n(s,a) : number of times action a taken in s
  - n(s) : number of times state s encountered

$$\pi_{UCT}(s) = \arg\max_a Q(s,a) + c\sqrt{\frac{\ln n(s)}{n(s,a)}}$$

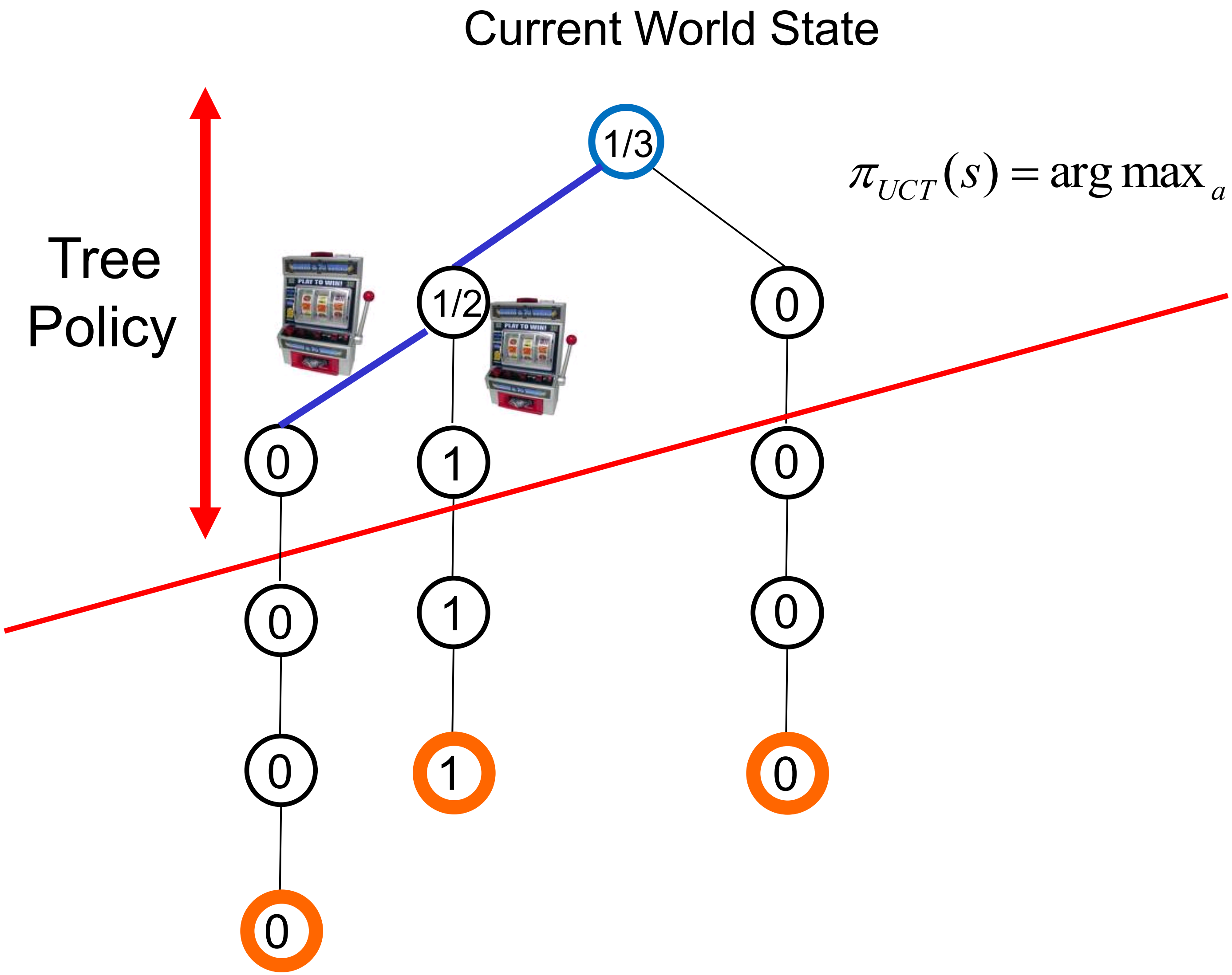
Theoretical constant that must be selected empirically in practice

When all node actions tried once, select action according to tree policy



$$\pi_{UCT}(s) = \arg\max_a Q(s,a) + c\sqrt{\frac{\ln n(s)}{n(s,a)}}$$

When all node actions tried once, select action according to tree policy



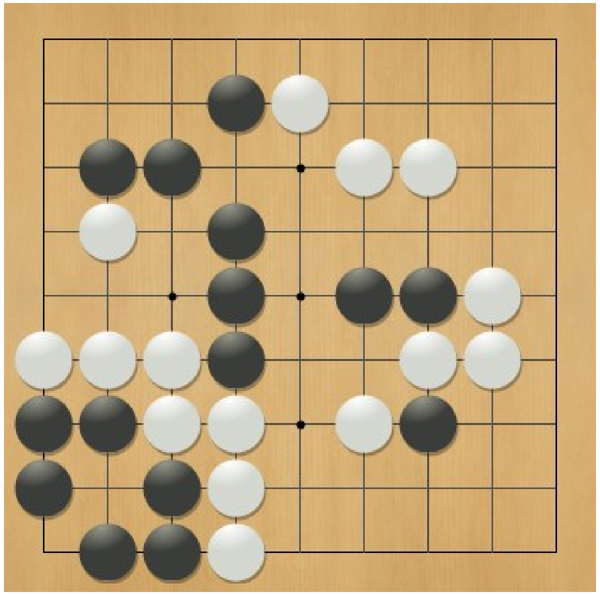
$$\pi_{UCT}(s) = \arg\max_a Q(s,a) + c\sqrt{\frac{\ln n(s)}{n(s,a)}}$$

UCT Recap

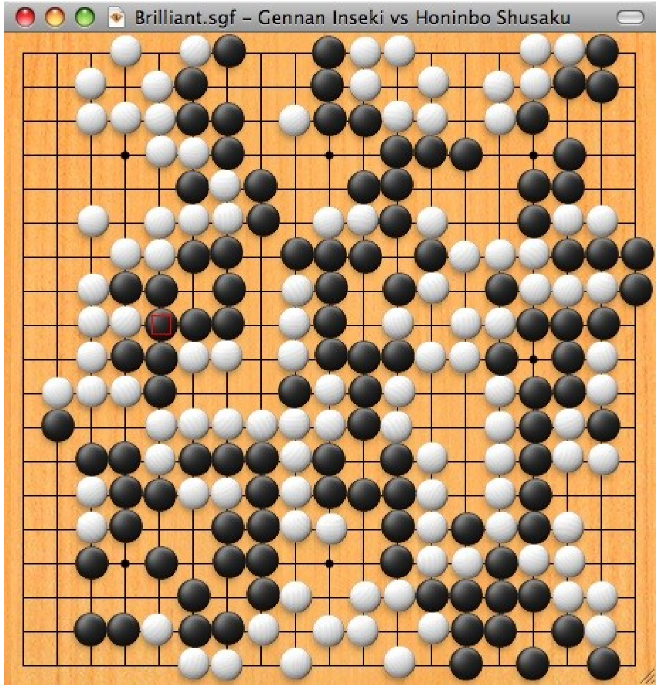
- To select an action at a state s
  - Build a tree using N iterations of monte-carlo tree search
    - Default policy is uniform random
    - Tree policy is based on UCB rule
  - Select action that maximizes Q(s,a)
    (note that this final action selection does not take the exploration term into account, just the Q-value estimate)

- The more simulations the more accurate

Computer Go



9x9 (smallest board)



19x19 (largest board)

- “Task Par Excellence for AI” (Hans Berliner)
- “New Drosophila of AI” (John McCarthy)
- “Grand Challenge Task” (David Mechner)

A Brief History of Computer Go

- *2005*: Computer Go is impossible!
- *2006*: UCT invented and applied to 9x9 Go *(Kocsis, Szepesvari; Gelly et al.)*
- *2007*: Human master level achieved at 9x9 Go *(Gelly, Silver; Coulom)*
- *2008*: Human grandmaster level achieved at 9x9 Go *(Teytaud et al.)*

Computer GO Server: 1800 ELO → 2600 ELO

Other Successes

- Klondike Solitaire (wins 40% of games)
- General Game Playing Competition
- Real-Time Strategy Games
- Combinatorial Optimization

- List is growing

- Usually extend UCT is some ways

Some Improvements

- Use domain knowledge to handcraft a more intelligent default policy than random
  - E.g. don't choose obviously stupid actions

- Learn a heuristic function to evaluate positions
  - Use the heuristic function to initialize leaf nodes (otherwise initialized to zero)

Summary

- When you have a tough planning problem and a simulator
  - Try Monte-Carlo planning
- Basic principles derive from the multi-arm bandit
- Policy Rollout is a great way to exploit existing policies and make them better
- If a good heuristic exists, then shallow sparse sampling can give good gains
- UCT is often quite effective especially when combined with domain knowledge