On the Complexity of Bandit and Derivative-Free Stochastic Convex Optimization

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Bandit / Derivative-Free Stochastic Convex Optimization

Setting

- Convex domain $\mathcal{W} \subseteq \mathbb{R}^d$
- Unknown convex function $F : \mathcal{W} \mapsto \mathbb{R}$
- Can get $F(\mathbf{w})$+noise at any $\mathbf{w} \in \mathcal{W}$
- Want to optimize F with as few queries as possible

Information is *zeroth-order*: No direct access to gradients/Hessians

Bandit / Derivative-Free Stochastic Convex Optimization

Optimization Community

- Derivative-Free / Zeroth-Order SCO
- Black-box situations where gradient is hard to compute / unavailable
- Goal: Minimize *optimization error*

$$\mathbb{E}\left[F(\bar{\mathbf{w}}_T) - F(\mathbf{w}^*)\right]$$

Online Learning Community

- Bandit SCO
- Sequential decision making under uncertainty (e.g. multi-armed bandits)
- Goal: Minimize *regret*

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} F(\mathbf{w}_t) - F(\mathbf{w}^*)\right]$$

Minimizing regret is harder than minimizing error

Attainable Performance

What is the attainable error/regret in terms of

- Number of queries T
- Dimension d

With gradient information, situation is simple:

	Error		Regret	
Function Type	$\mathcal{O}$	Ω	$\mathcal{O}$	Ω
Convex	$\sqrt{1/T}$			
Strongly Convex	$1/T$		$\log(T)/T$	

[Zinkevich 2003],[Hazan et al. 2006],[Hazan and Kale 2011]

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- Linear F, other convex domains:
  $\sqrt{d/T}$ to $\sqrt{d^2/T}$ bounds, crucially depending on domain

  [Abbasi-Yadkori et al. 2011],[Audibert et al. 2011],[Bubeck et al. 2012]

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- General convex F:
  - $\mathcal{O}\left(\left(d^2/T\right)^{1/4}\right)$ [Flaxman et al. 2005]
  - $\mathcal{O}\left(\sqrt{d^{34}/T}\right)$ [Agarwal et al. 2011]

Yudin and Nemirosvki 1979

...Each of the methods suggested is in some respect unimprovable, but bad in other respects... The situation with a zeroth-order oracle is far from clear.

Our Results

We study the complexity of nonlinear bandit/derivative-free SCO - particularly strongly convex functions

1st Main Result

For strongly-convex and smooth functions, $\Theta(\sqrt{d^2/T})$ error/regret

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For strongly-convex and smooth functions, $\Theta(\sqrt{d^2/T})$ error/regret

- Follows from a new lower bound
- First tight complexity characterization for a general nonlinear class, with minimal/no domain assumptions
- Price of bandit information is quadratic in dimension d
- Stronger lower bound for strongly convex and convex functions

Our Results

2nd Main Result

In the special case of quadratic functions, attainable **error** is exactly $\Theta(d^2/T)$

- Improvable to $\mathcal{O}(d/T)$ under additional assumptions

- Demonstrates "fast rate" is possible even without gradient knowledge
- "Contradicts" $\Omega(\sqrt{d/T})$ lower bound presented in NIPS 2012 [Jamieson et al. 2012]

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3rd Main Result

Even for quadratic functions, attainable **regret** is $\Theta(\sqrt{d^2/T})$

- Much worse than attainable error

Our Results

	Error		Regret	
Function Type	$\mathcal{O}$	Ω	$\mathcal{O}$	Ω
Quadratic	$\mathbf{\frac{d^2}{T}}$		$\sqrt{\mathbf{\frac{d^2}{T}}}$	
St. Convex + Smooth	$\sqrt{\mathbf{\frac{d^2}{T}}}$			
Str. Convex	$\min\left\{\sqrt[3]{\frac{d^2}{T}}, \sqrt{\frac{d^{34}}{T}}\right\}$	$\sqrt{\mathbf{\frac{d^2}{T}}}$	$\min\left\{\sqrt[3]{\frac{d^2}{T}}, \sqrt{\frac{d^{34}}{T}}\right\}$	$\sqrt{\mathbf{\frac{d^2}{T}}}$
Convex	$\min\left\{\sqrt[4]{\frac{d^2}{T}}, \sqrt{\frac{d^{34}}{T}}\right\}$	$\sqrt{\mathbf{\frac{d^2}{T}}}$	$\min\left\{\sqrt[4]{\frac{d^2}{T}}, \sqrt{\frac{d^{34}}{T}}\right\}$	$\sqrt{\mathbf{\frac{d^2}{T}}}$

Quadratic Functions: Upper Bounds

Special case of strongly-convex and smooth functions

$$F(\mathbf{w}) = \mathbf{w}^\top A\mathbf{w} + \mathbf{b}^\top \mathbf{w} + c$$

- A is positive-definite ($\lambda := \lambda_{\min}(A) > 0$)
- Scaled so that $\|A\|, \|\mathbf{b}\|, |c| \leq 1$

Assumption: we can query in a ball of radius ϵ around optimum $\mathbf{w}^*$

Quadratic Functions: Upper Bounds

Algorithm

Input: $\lambda, \epsilon > 0$
Initialize $\mathbf{w}_1 = \mathbf{0}$.
for $t = 1, \ldots, T-1$ **do**
  Pick $\mathbf{r} \in \{-1, +1\}^d$ uniformly at random
  Query noisy function value v at point $\mathbf{w}_t + \frac{\epsilon}{\sqrt{d}}\mathbf{r}$
  Let $\tilde{\mathbf{g}} = \frac{\sqrt{d}v}{\epsilon}\mathbf{r}$
  Let $\mathbf{w}_{t+1} = \Pi_{\bar{\mathcal{W}}}\left(\mathbf{w}_t - \frac{1}{\lambda t}\tilde{\mathbf{g}}\right)$
end for
Return $\bar{\mathbf{w}}_T = \frac{2}{T}\sum_{t=T/2}^{T}\mathbf{w}_t$.

Key Obervation: For quadratic functions, 1-point Gradient estimate is unbiased even if query far from $\mathbf{w}_t$

Quadratic Functions: Upper Bounds

Theorem

If $\mathbf{w}^*$ has constant norm, then

$$\mathbb{E}\left[F(\bar{\mathbf{w}}_T) - F(\mathbf{w}^*)\right] \leq \mathcal{O}\left(\frac{1}{\epsilon^2}\frac{d^2}{\lambda T}\right)$$

- Jamieson et al. (NIPS 2012) show $\Omega(\sqrt{d/T})$ lower bound for such quadratic functions
- However, their domain shrinks with T, implying $\epsilon \to 0$, while we assume ϵ is fixed

Quadratic Functions: Upper Bounds

Aside: Result is improvable in some cases:

Theorem

Suppose $F(\mathbf{w}) = R(\mathbf{w}) + \mathbb{E}\left[\mathbf{w}^\top \hat{A}\mathbf{w} + \hat{\mathbf{b}}^\top \mathbf{w} + \hat{c}\right]$, where

- $R(\mathbf{w})$ is a known strongly convex function
- Queries are based on noisy realizations of $\hat{A}, \hat{\mathbf{b}}, \hat{c}$, and can be anywhere in $\mathbb{R}^d$

Then $\exists$ algorithm such that $\mathbb{E}\left[F(\bar{\mathbf{w}}_T) - F(\mathbf{w}^*)\right] \leq \mathcal{O}\left(\frac{d\ \mathbb{E}\left[\|\hat{A}\|_F^2\right]}{\lambda T}\right)$

Example (Ridge Regression)

$$F(\mathbf{w}) = \frac{\lambda}{2}\|\mathbf{w}\|^2 + \mathbb{E}\left[(\langle \mathbf{w}, \mathbf{x}\rangle - y)^2\right]$$

$$\hat{A} = \mathbf{x}\mathbf{x}^\top \quad \Rightarrow \quad \mathbb{E}\left[\|\hat{A}\|_F^2\right] \leq \mathcal{O}(1) \quad \Rightarrow \quad \text{error } \mathcal{O}\left(\frac{d}{\lambda T}\right)$$

Theorem

$\forall$ querying strategy, $\exists$ quadratic F (1-strongly convex, Lipschitz, minimized within unit Euclidean ball) such that

$$\mathbb{E}\left[F(\bar{\mathbf{w}}_T) - F(\mathbf{w}^*)\right] \geq \Omega\left(\frac{d^2}{T}\right)$$

Result holds even if can query anywhere in $\mathbb{R}^d$ (under reasonable noise assumptions)

Proof Idea

Adversary Strategy:

- Pick $\mathbf{e}$ uniformly at random from $\Theta\left(\sqrt{\frac{d}{T}}\right) \times \{-1,+1\}^d$
- $F(\mathbf{w}) = \frac{1}{2}\|\mathbf{w}\|^2 - \langle \mathbf{e}, \mathbf{w} \rangle$
  - Minimized at $\mathbf{e}$
- Given query point $\mathbf{w}$, return $F(\mathbf{w}) + \xi$

Key Idea:

- Due to strong convexity, suboptimality reduced to a **sum of** d hypothesis testing problems:

$$\mathbb{E}\left[F(\bar{\mathbf{w}}_T) - F(\mathbf{w}^*)\right] \geq \mathbb{E}\left[\frac{1}{2}\|\bar{\mathbf{w}}_T - \mathbf{e}\|^2\right] \geq \mathbb{E}\left[\Theta\left(\frac{d}{T}\right) \times \sum_{i=1}^{d} \mathbf{1}_{\bar{w}_i e_i < 0}\right]$$

- Result derived from a relative-entropy argument
  - # samples needed to distinguish sign(e_i)

Quadratic Functions: Lower Bounds

Theorem

Under same conditions as above, $\forall$ querying strategy, $\exists$ quadratic F such that $\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} F(\mathbf{w}_t) - F(\mathbf{w}^*)\right] \geq \Omega\left(\sqrt{\frac{d^2}{T}}\right)$

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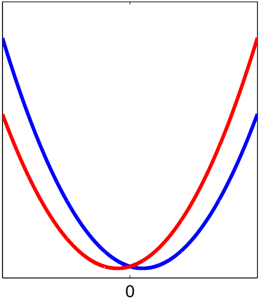
Key Idea

- With more careful analysis, relative entropy terms depend on $\|\mathbf{w}_t\|$
- For small regret, $\mathbf{w}_t$ must be close to optimum $\mathbf{e}$
- If $\|\mathbf{e}\|$ small $\Rightarrow$ $\|\mathbf{w}_t\|$ must be small $\Rightarrow$ Larger lower bound



Strongly Convex and Smooth Functions

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$\forall$ querying strategy, $\exists$ strongly-convex and smooth F (minimized within unit Euclidean ball) such that

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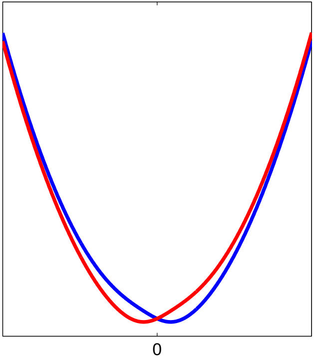
Key Idea

- $F(\mathbf{w}) = \|\mathbf{w}\|^2 - \sum_{i=1}^{d} \frac{e_i w_i}{1+(w_i/e_i)^2}$
  - $\mathbf{e}$ again selected at random
- $F(\mathbf{w}) \approx \|\mathbf{w}\|^2 - 0.9 \langle \mathbf{e}, \mathbf{w} \rangle$ near optimum, but $F(\mathbf{w}) \approx \|\mathbf{w}\|^2$ further away

  $\Rightarrow$ Querying far from optimum doesn't give information on $\mathbf{e}$
- $\Rightarrow$ Same $\sqrt{d^2/T}$ lower bound as for *regret* in quadratic case



Summary

- **Exact characterization of bandit/derivative-free SCO for strongly-convex and smooth functions**
  - Implies new lower bounds for more general settings
  - Quadratic dependence on the dimension is inevitable
- **"Fast" error rate achievable even without gradients**, for quadratic functions
- **Substantial gaps** between optimization error and regret
- **Open questions:** Complexity of strongly convex (non-smooth) problems? General convex problems?
  - $\Omega\left(\sqrt{d^2/T}\right)$ vs. $\mathcal{O}\left(\min\left\{\sqrt{\frac{d^{34}}{T}}, \left(\frac{d^2}{T}\right)^{1/4}\right\}\right)$
  - Conjecture: $\Theta\left(\sqrt{d^2/T}\right)$, but **need new algorithms!**