# Connections between the Lasso and Support Vector Machines

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*Ecole Polytechnique*

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# Outline

- An Equivalence between
  the *Lasso* and *Support Vector Machines*
    - Reduction from Lasso to SVM
    - Reduction from SVM to Lasso
    - Applications
- Greedy Algorithms
  (from optimization and signal processing)

# SVM

= large margin
linear ***classifier***

Training data

# SVM

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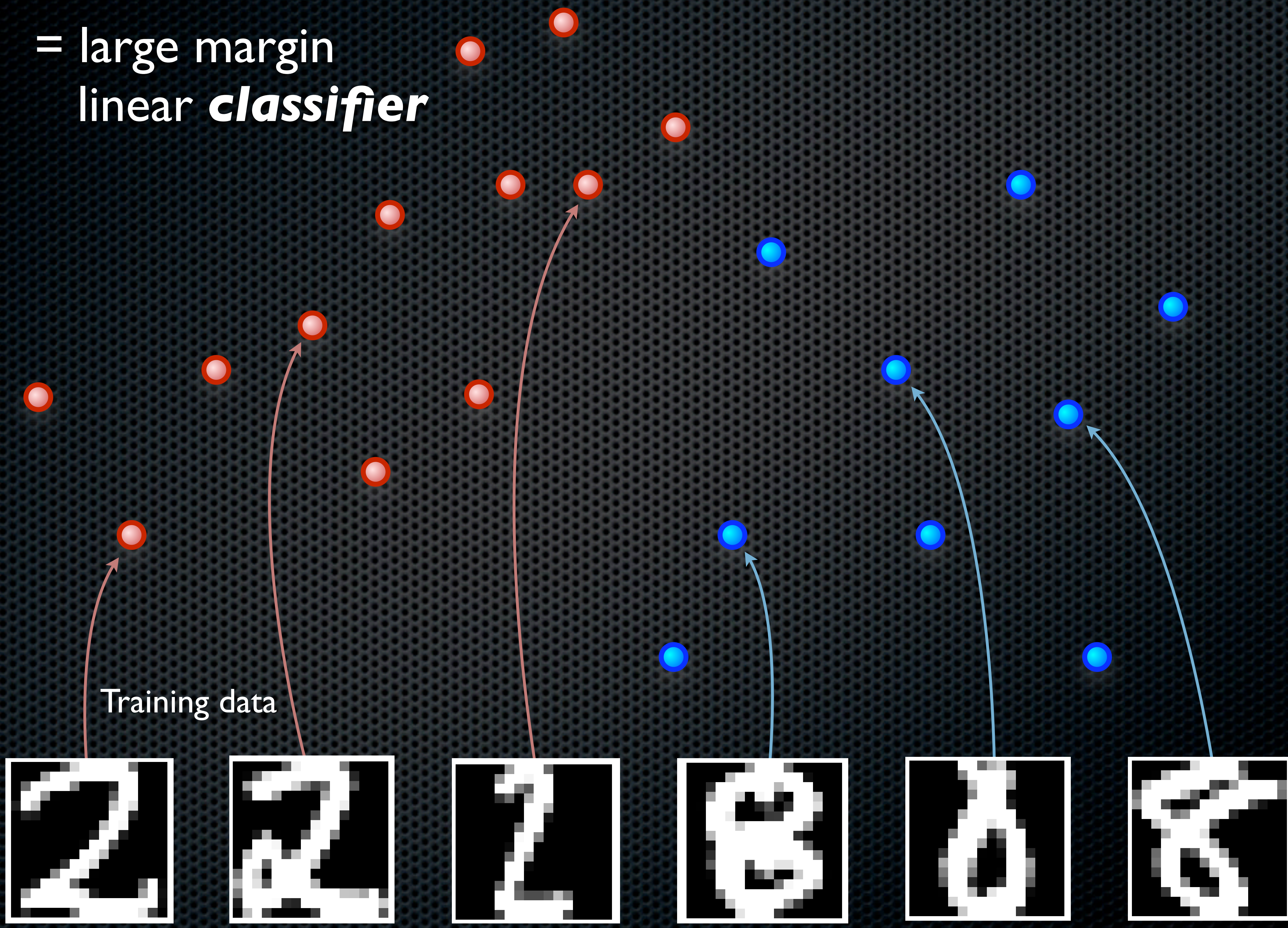

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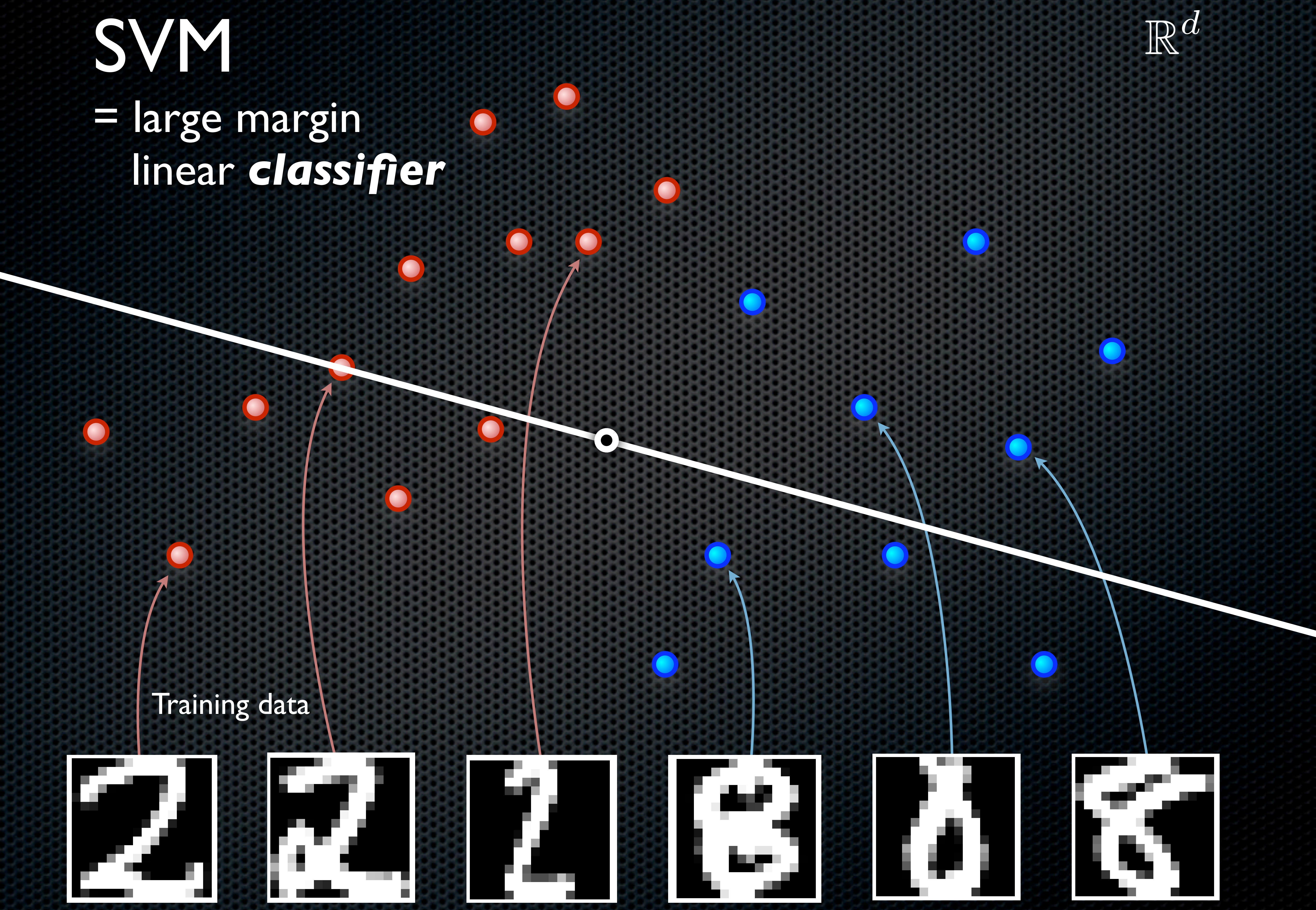

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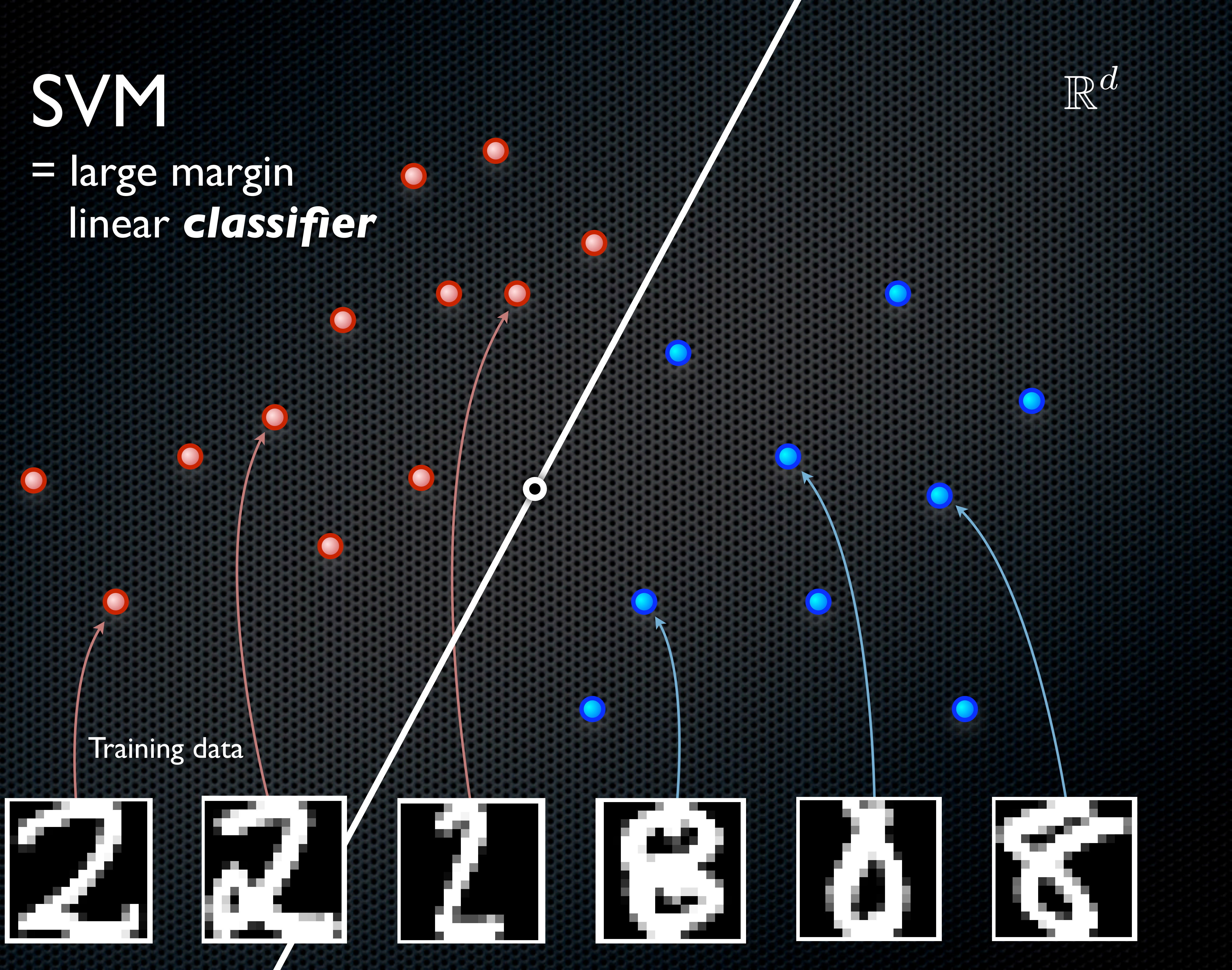

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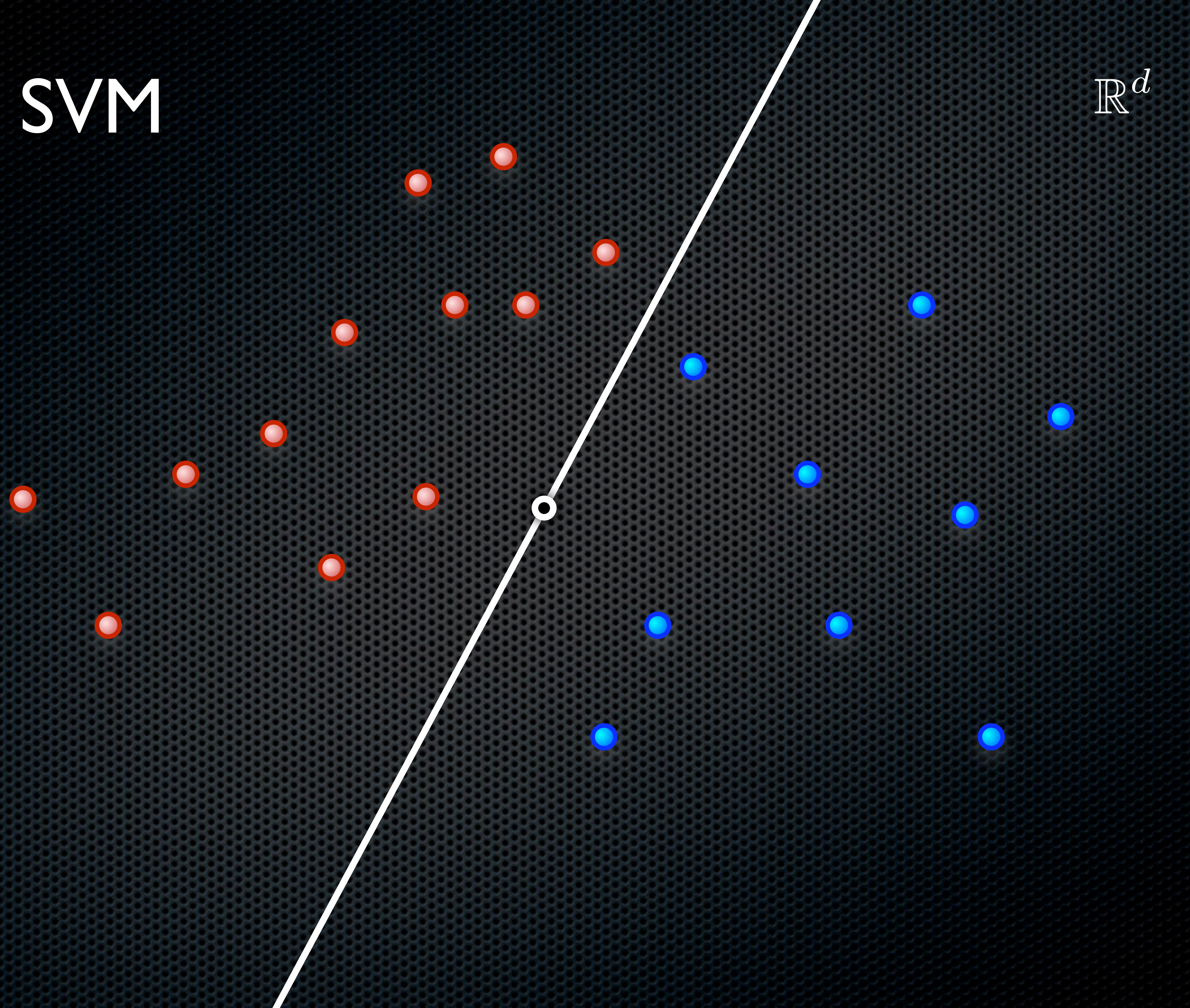

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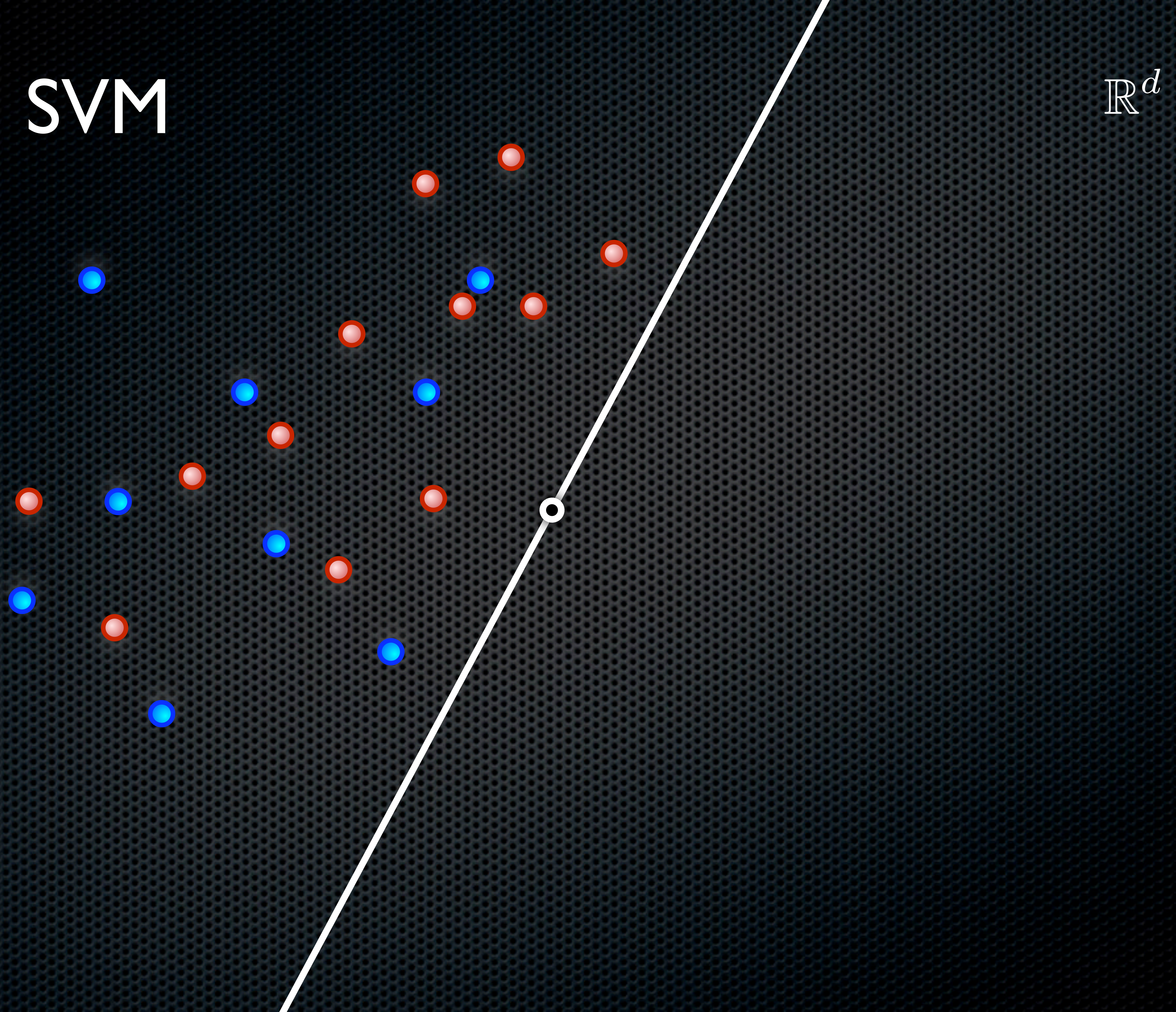

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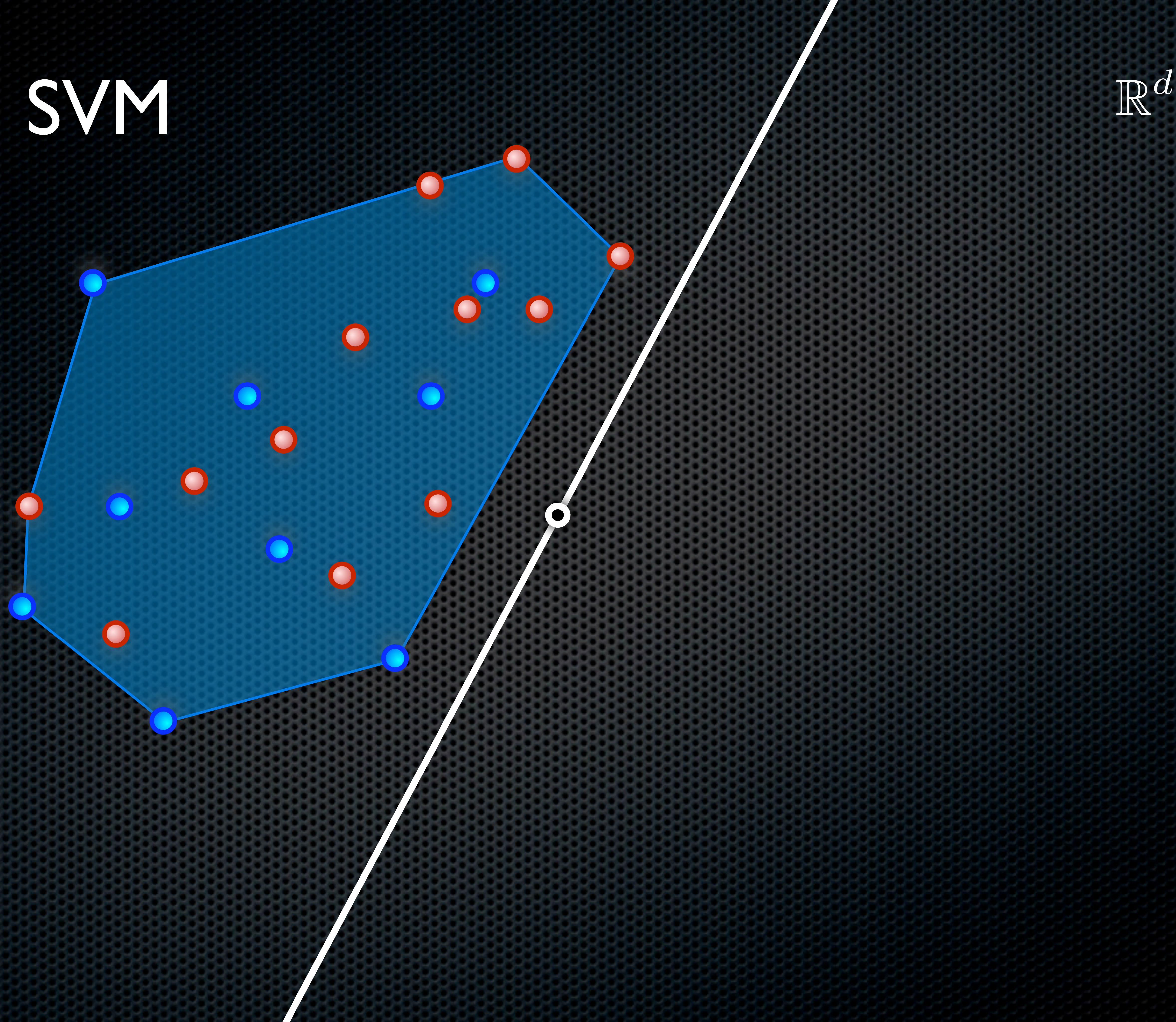
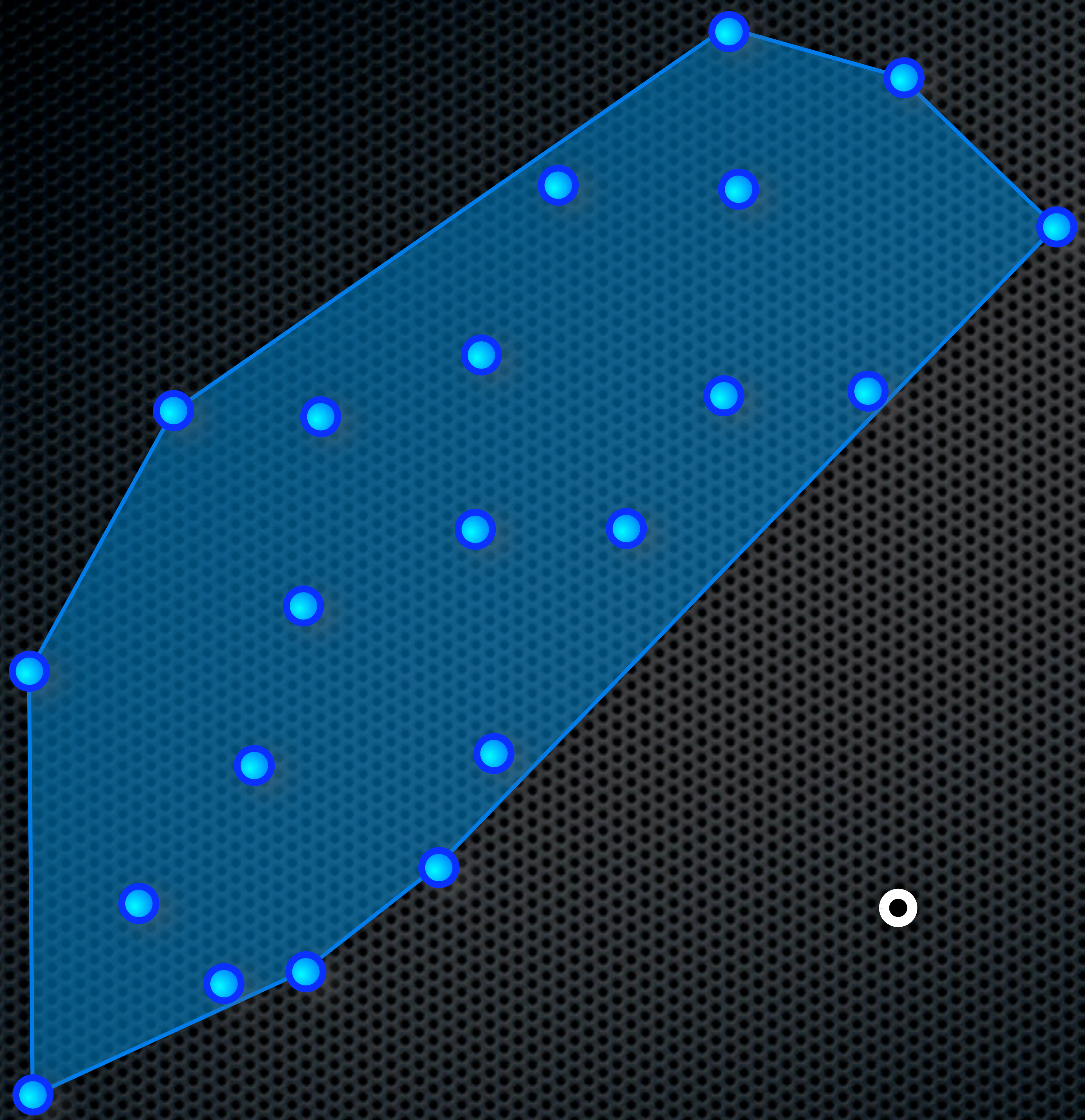
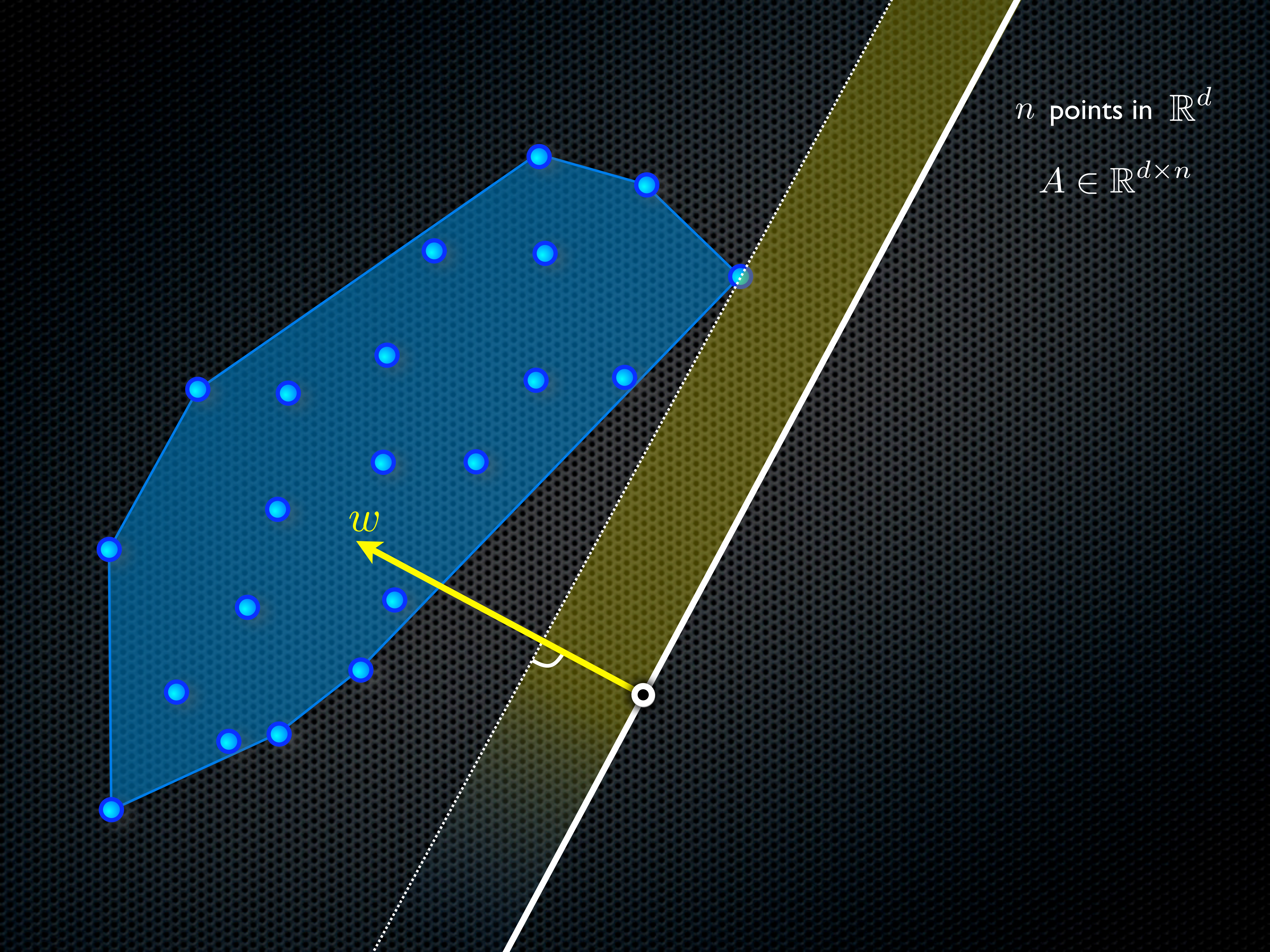




$n$ points in $\mathbb{R}^d$

$A \in \mathbb{R}^{d \times n}$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$w$

# Polytope distance

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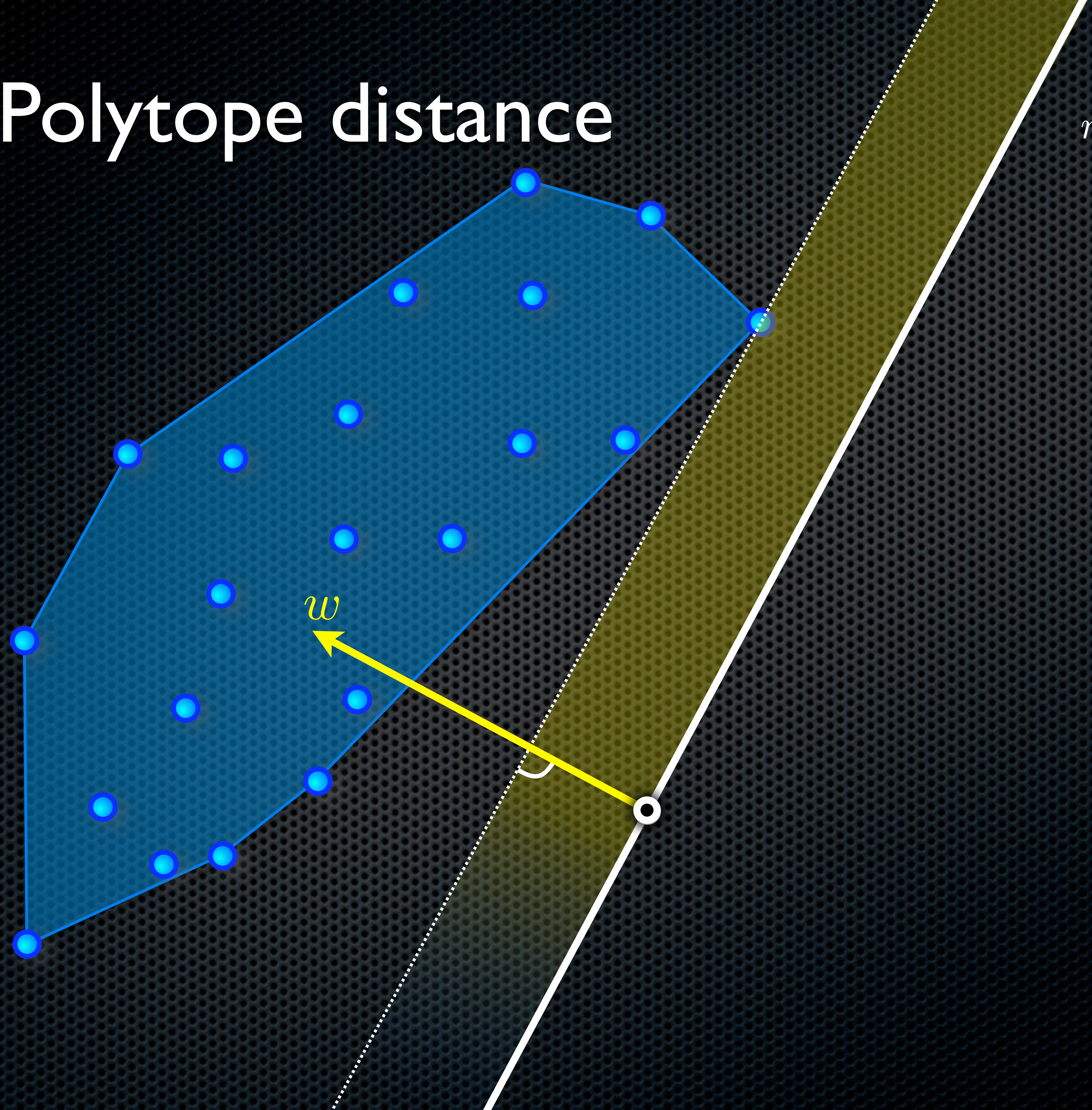

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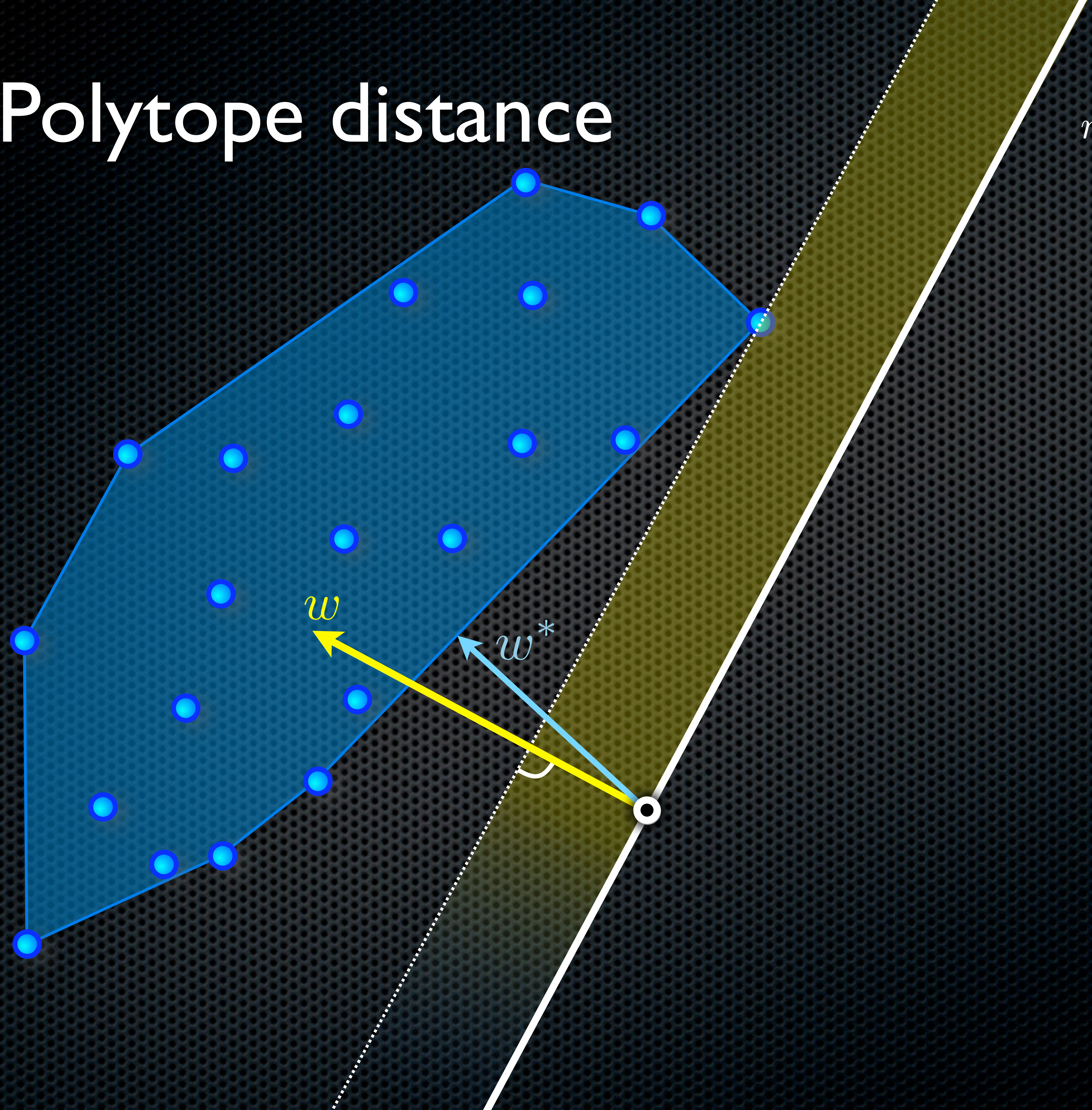

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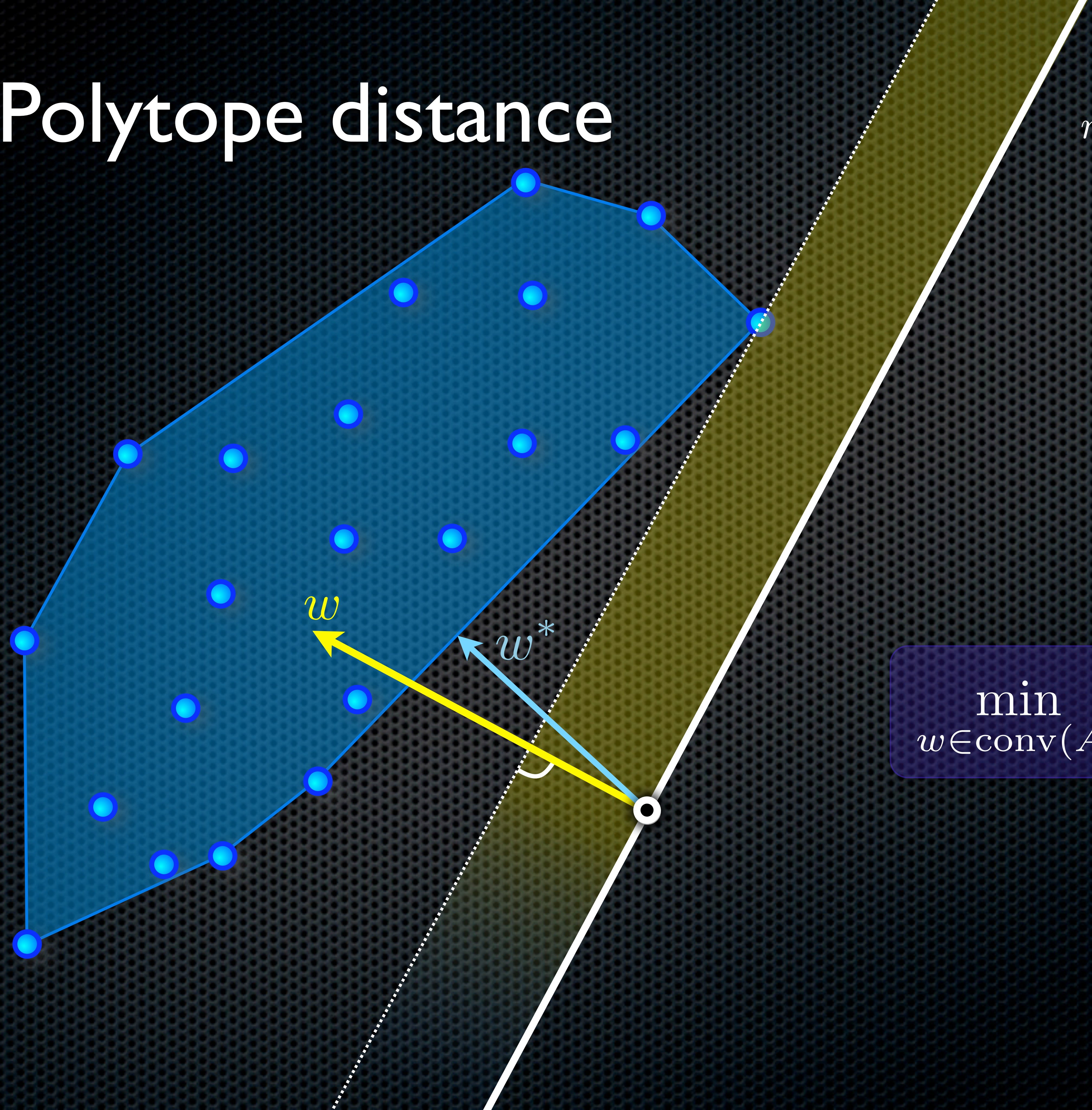


$$\min_{w \in \mathrm{conv}(A)} \|w\|^2$$

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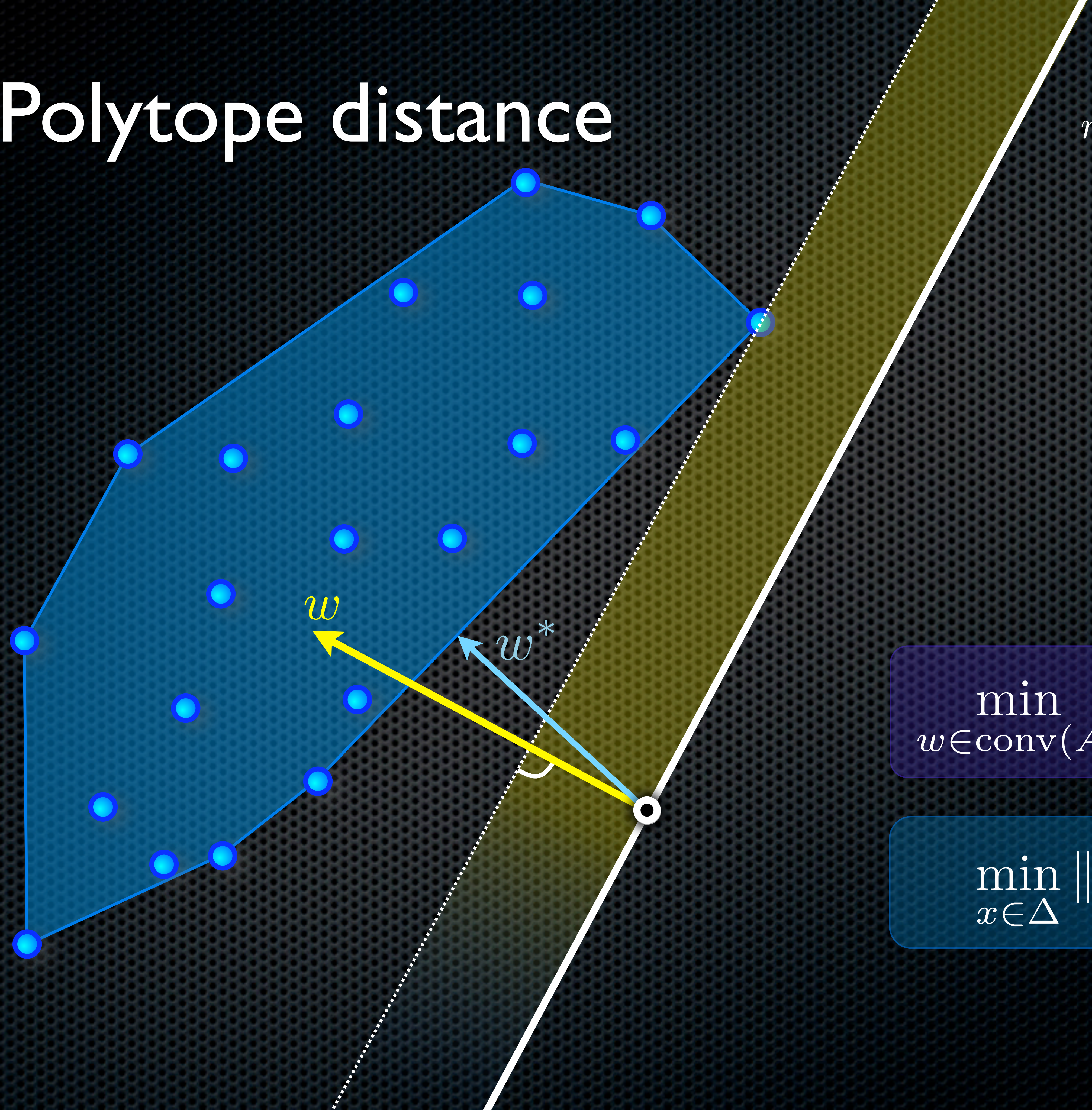


$$\min_{w \in \mathrm{conv}(A)} \|w\|^2$$

$$\min_{x \in \Delta} \|Ax\|^2$$

$A \in \mathbb{R}^{d \times n}$

# SVM variants

## whose *dual* problem is of the form $\min_{x \in \Delta} \|Ax\|^2$

| | ***Hard margin*** | ***Soft margin (L2-loss)*** | ***Soft margin (L1-loss)*** |
|---|---|---|---|
| ***Two class*** *no offset/bias* | ✓ | ✓ | ✗ |
| ***Two class*** *regularized offset/bias* | ✓ | ✓ | ✗ |
| ***One Class*** | ✓ | ✓ | ✗ |

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$$\min_{\substack{\bar{w}\in\mathbb{R}^d,\ \rho\in\mathbb{R},\\ \xi\in\mathbb{R}^n}} \quad \frac{1}{2}\|\bar{w}\|_2^2 - \rho + \frac{C}{2}\sum_i \xi_i^2$$
$$s.t. \quad y_i \cdot \bar{w}^T X_i \geq \rho - \xi_i \quad \forall i \in [1..n]$$

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(all with or without using ***kernels***)

$\|Ax\|^2 = x^T A^T A x$

$$\min_{\substack{\bar{w} \in \mathbb{R}^d,\ \rho \in \mathbb{R}, \\ \xi \in \mathbb{R}^n}} \quad \frac{1}{2} \|\bar{w}\|_2^2 - \rho + \frac{C}{2} \sum_i \xi_i^2$$

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$A \in \mathbb{R}^{d \times n}$
$b \in \mathbb{R}^{d}$

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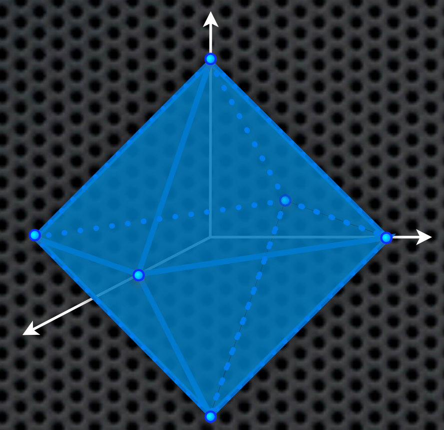
= $\ell_1$-regularized least squares ***regression***

$$\min_{\|x\|_1 \leq t} \|Ax - b\|^2$$



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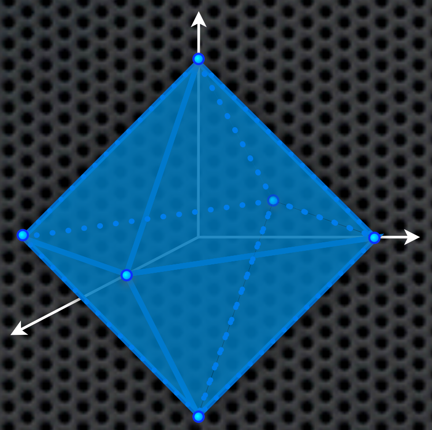

- Sparse regression

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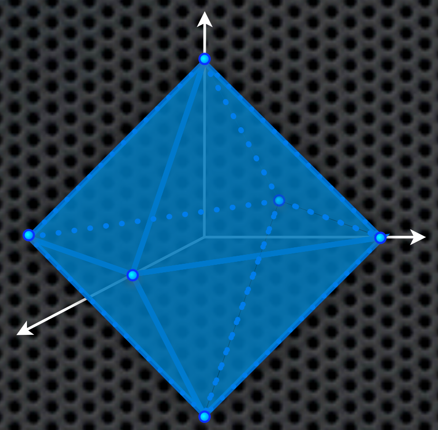

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- Feature selection

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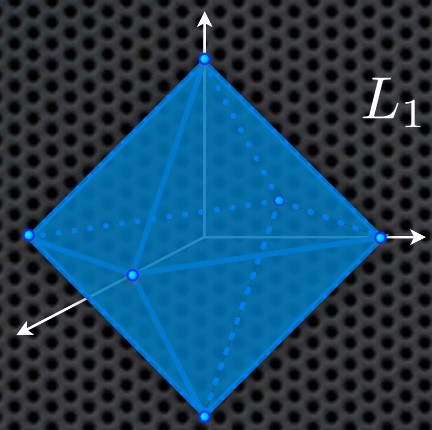

$$L_1 := \{x \in \mathbb{R}^n \mid \|x\|_1 \leq 1\}$$
$$= \mathrm{conv}(\{\pm \mathbf{e}_i\})$$

- Sparse regression
- Feature selection

$(\text{Lasso} \preceq \text{SVM})$

$A \in \mathbb{R}^{d \times n}$
$b \in \mathbb{R}^{d}$

Given a Lasso $\min_{x \in L_1} \|Ax - b\|^2$

construct an equivalent SVM instance $\min_{x' \in \Delta} \|\tilde{A}x'\|^2$

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$\in L_1 \subset \mathbb{R}^n$

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$x = \begin{bmatrix} \mathbf{I}_n & -\mathbf{I}_n \end{bmatrix} x'$

$x \in L_1 \subset \mathbb{R}^n$ $\quad$ $x' \in \Delta \subset \mathbb{R}^{2n}$

(***barycentric coordinates***)

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$$\min_{x'\in\Delta} \|A\begin{pmatrix} \mathbf{I}_n & -\,\mathbf{I}_n \end{pmatrix} x' - b\|^2$$

SVM:

$$\tilde{A} := \begin{pmatrix} A & -\,A \end{pmatrix} - b\mathbf{1}^T \in \mathbb{R}^{d\times 2n}$$

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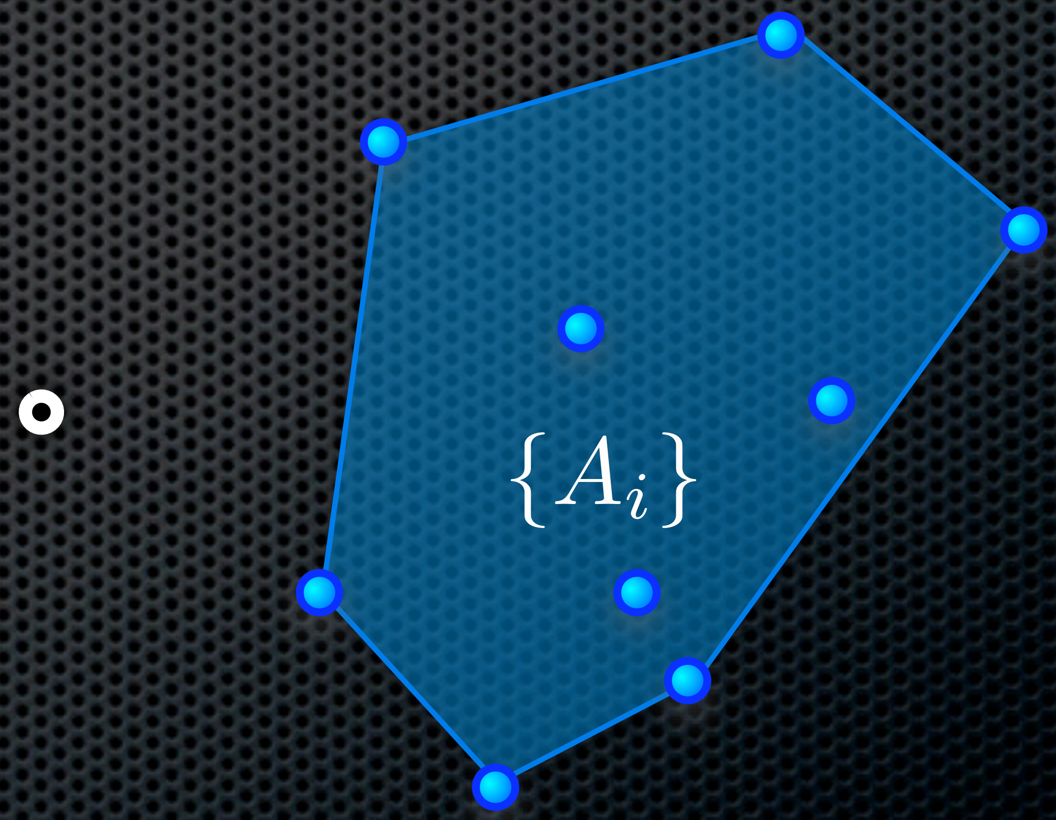
# Geometric interpretation:

$$\min_{x \in L_1} \|Ax - b\|^2$$



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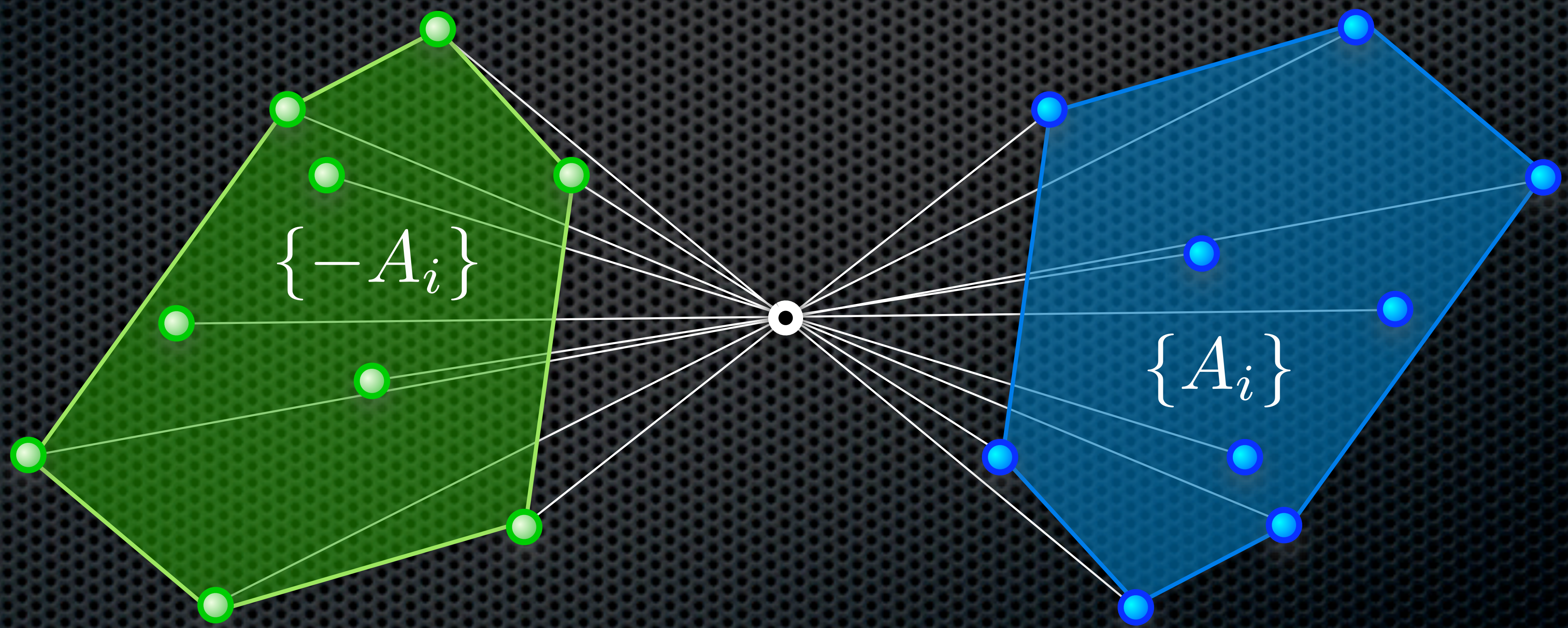

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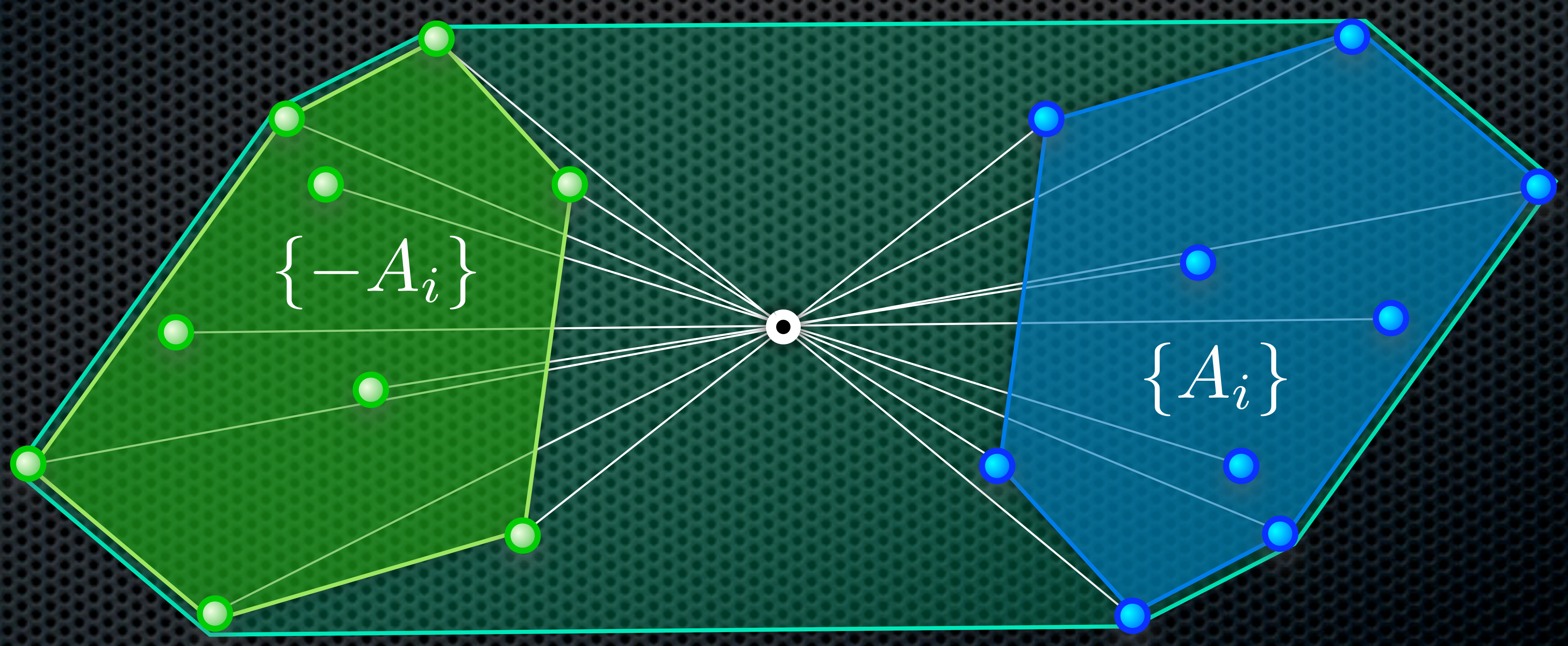

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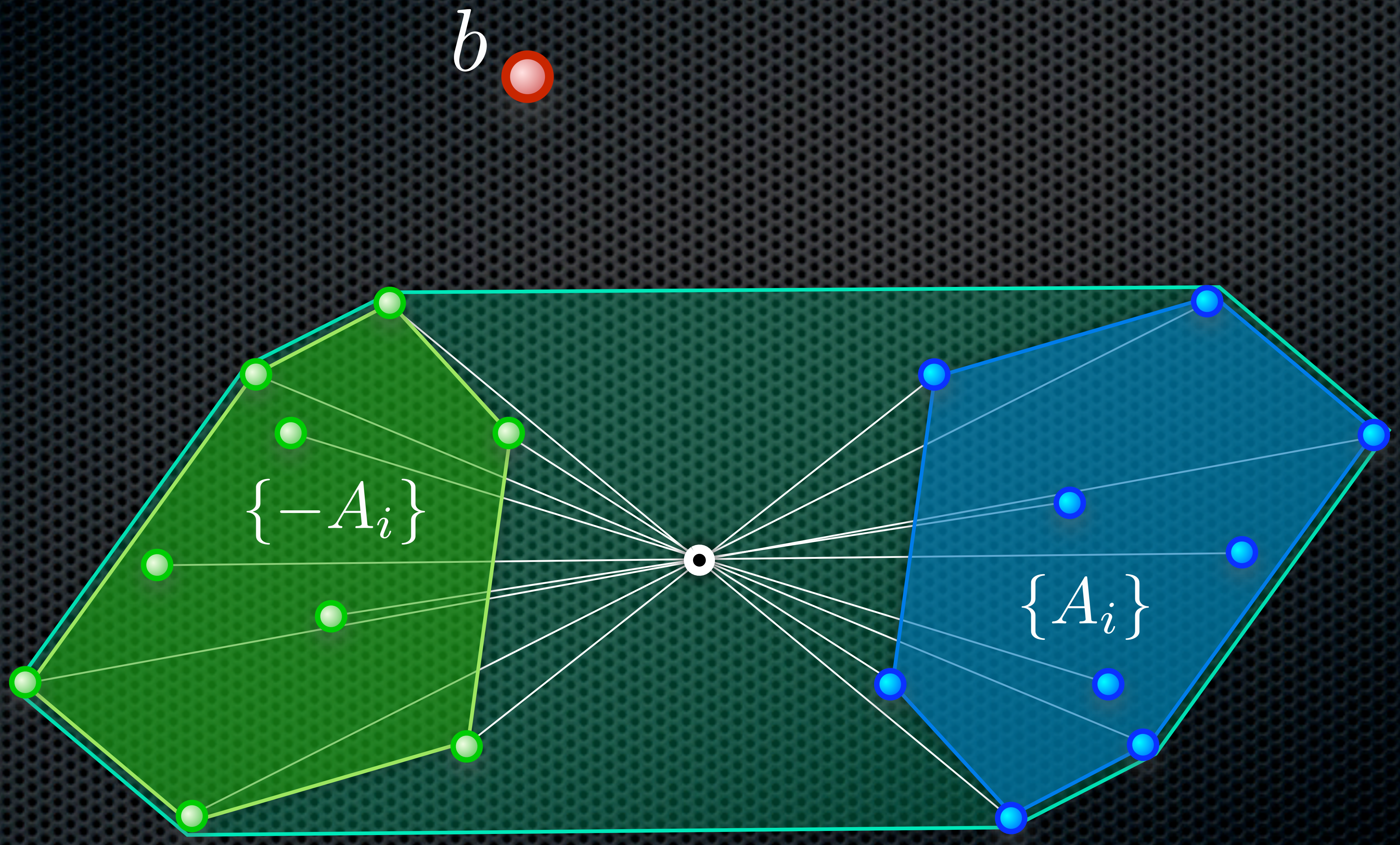

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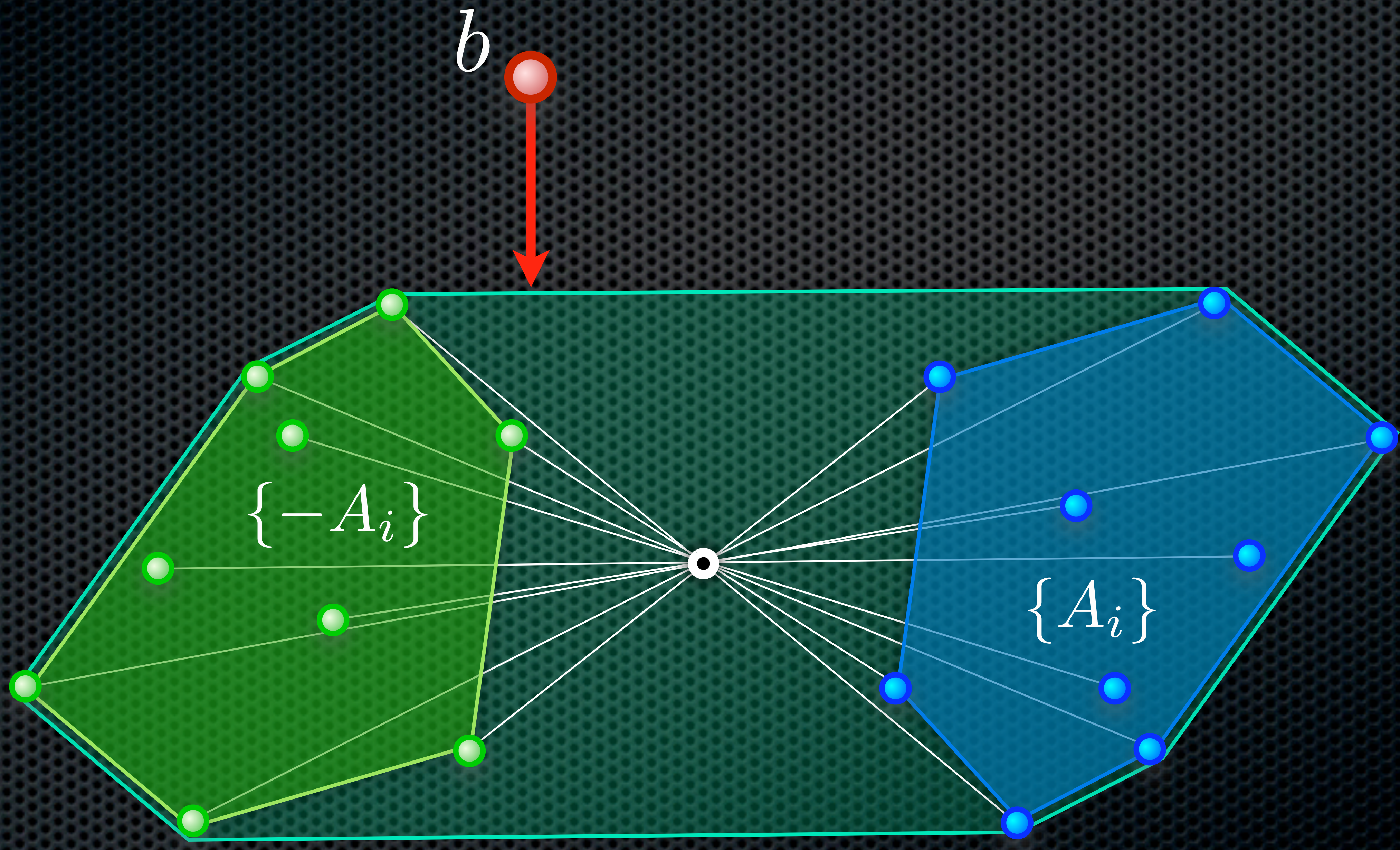

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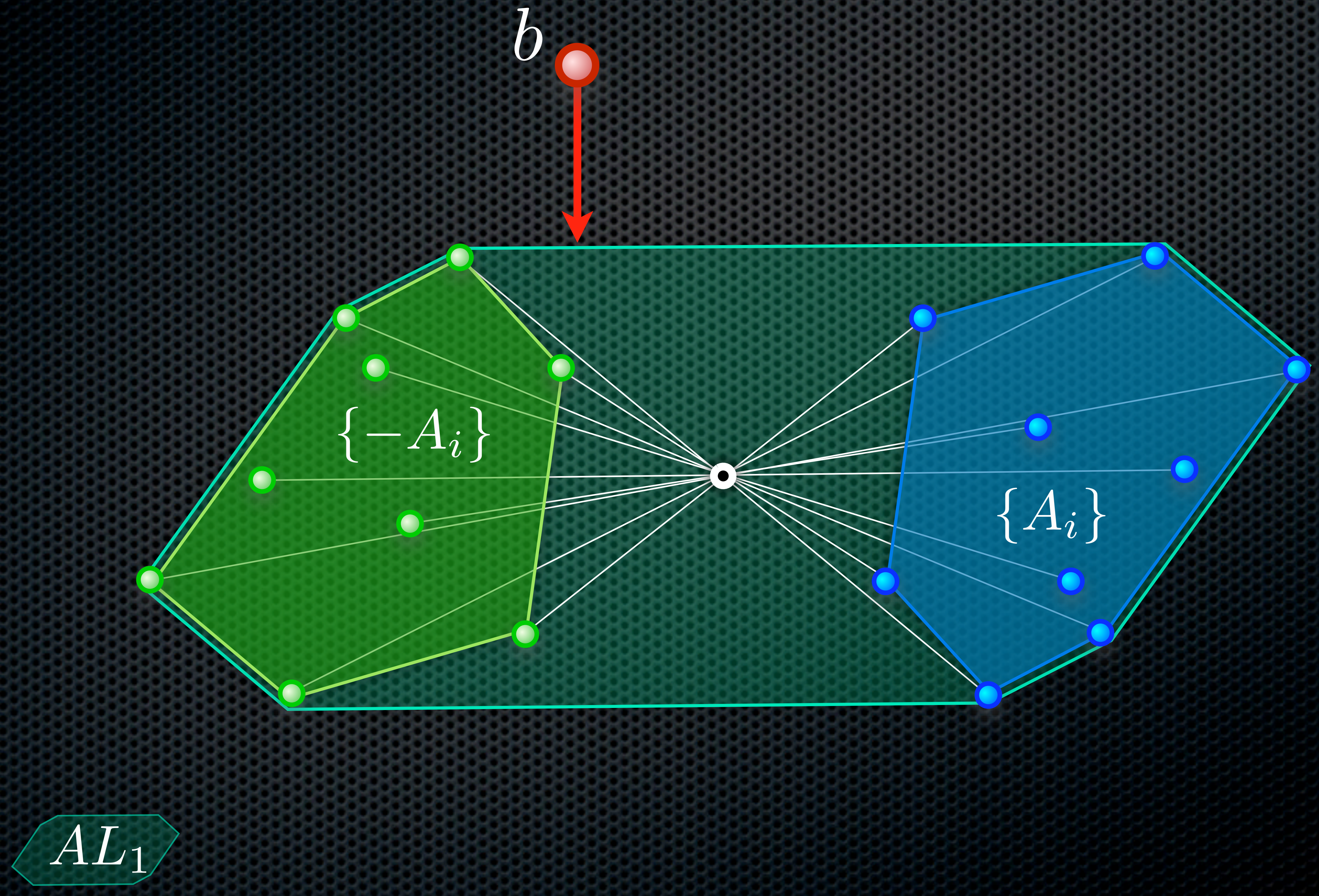

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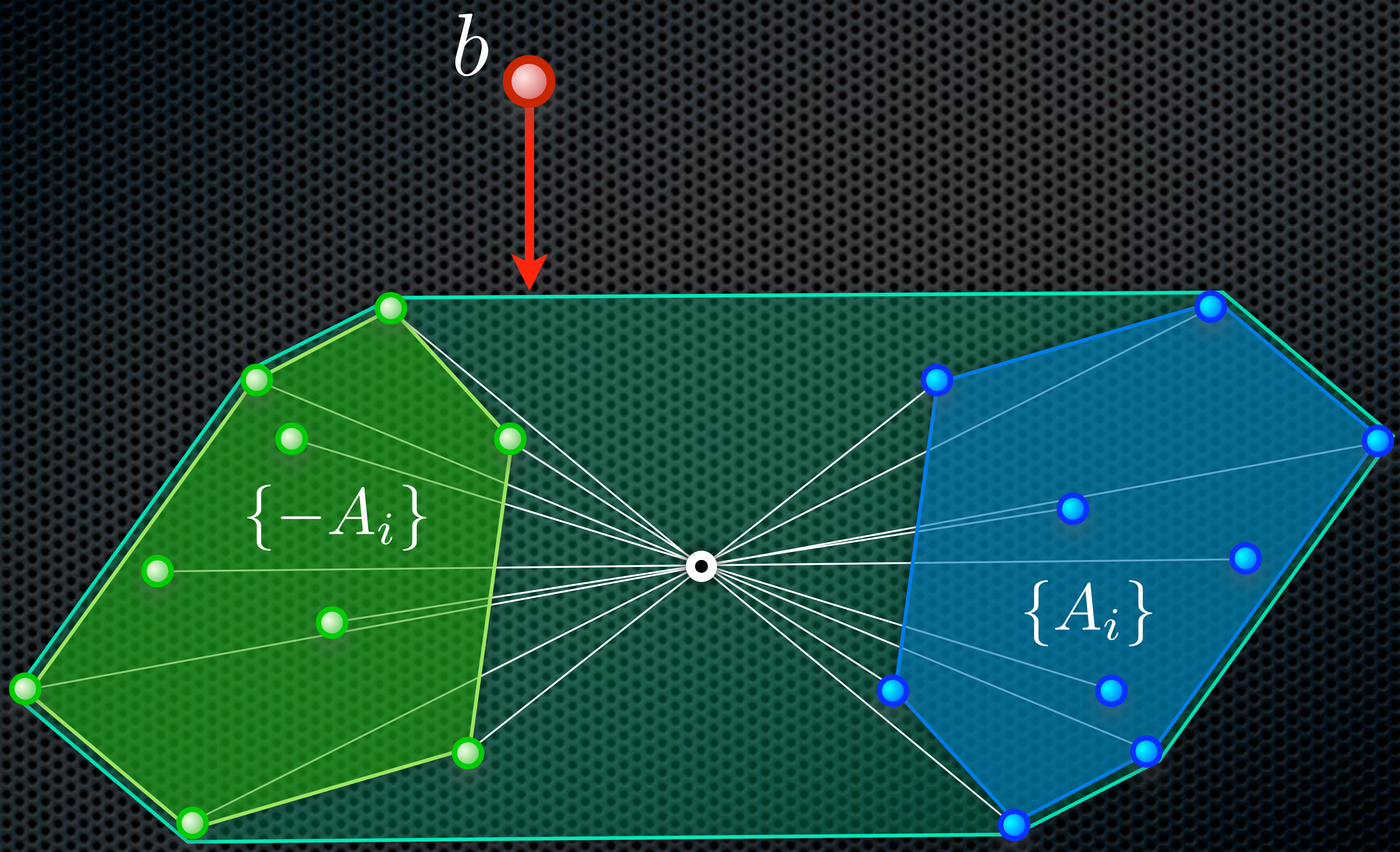


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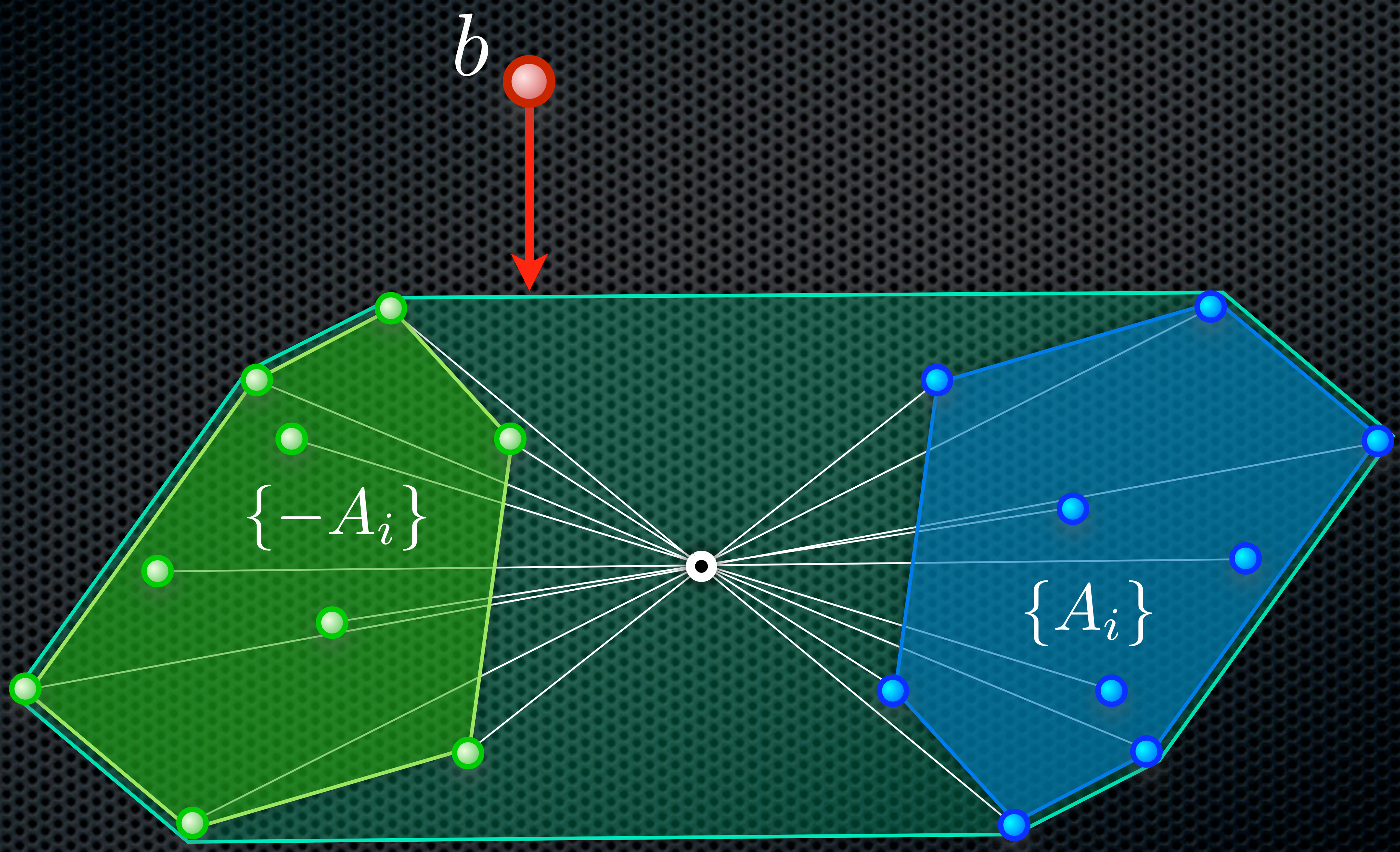


$A\,\mathrm{conv}(S)$
$= \mathrm{conv}(AS)$

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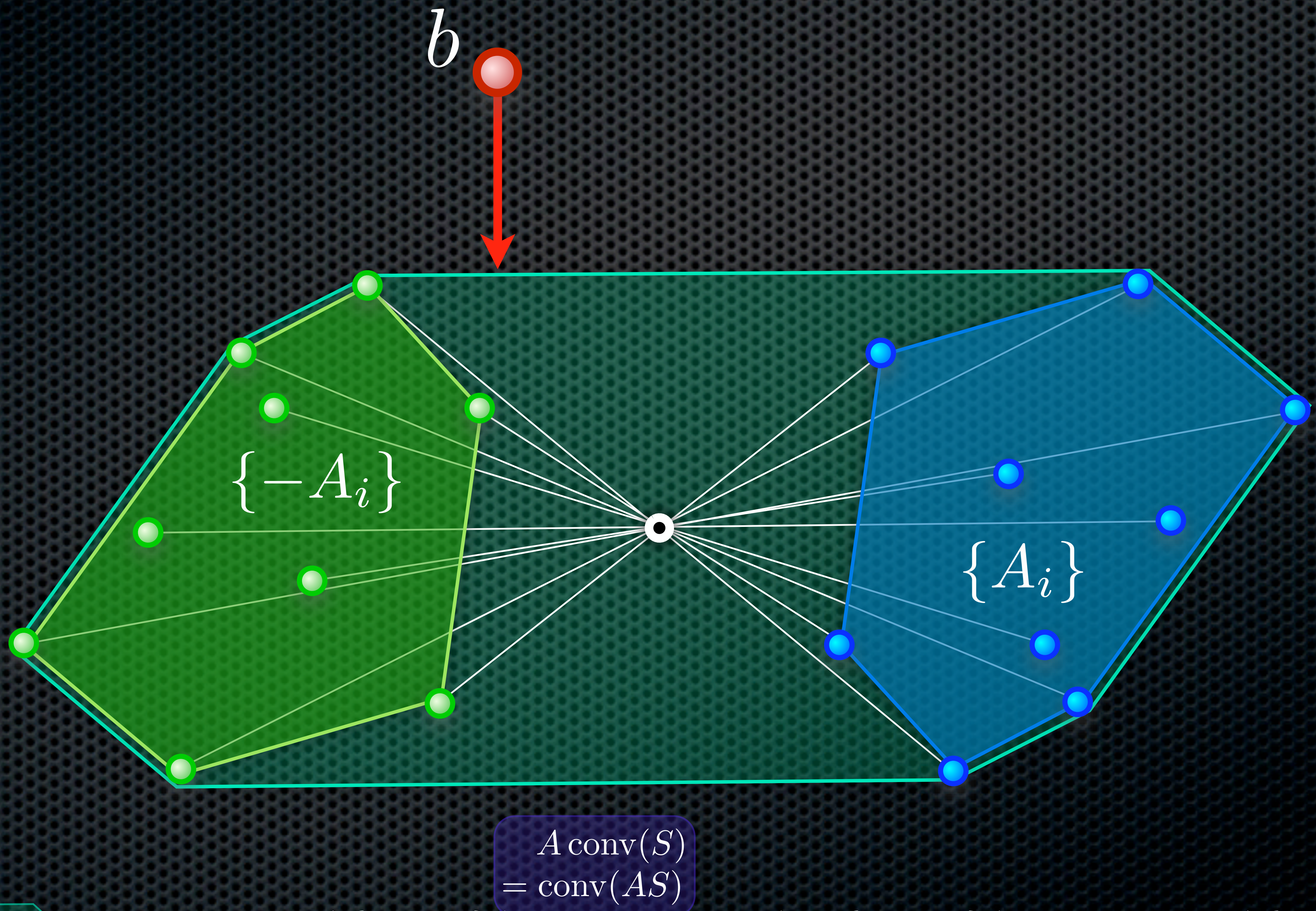


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more challenging reduction!

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$$\tilde{b} \propto -w$$

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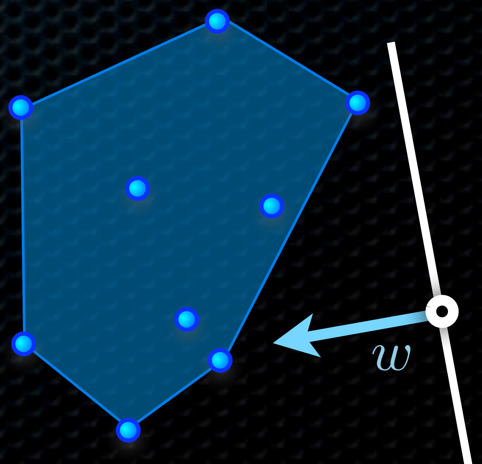

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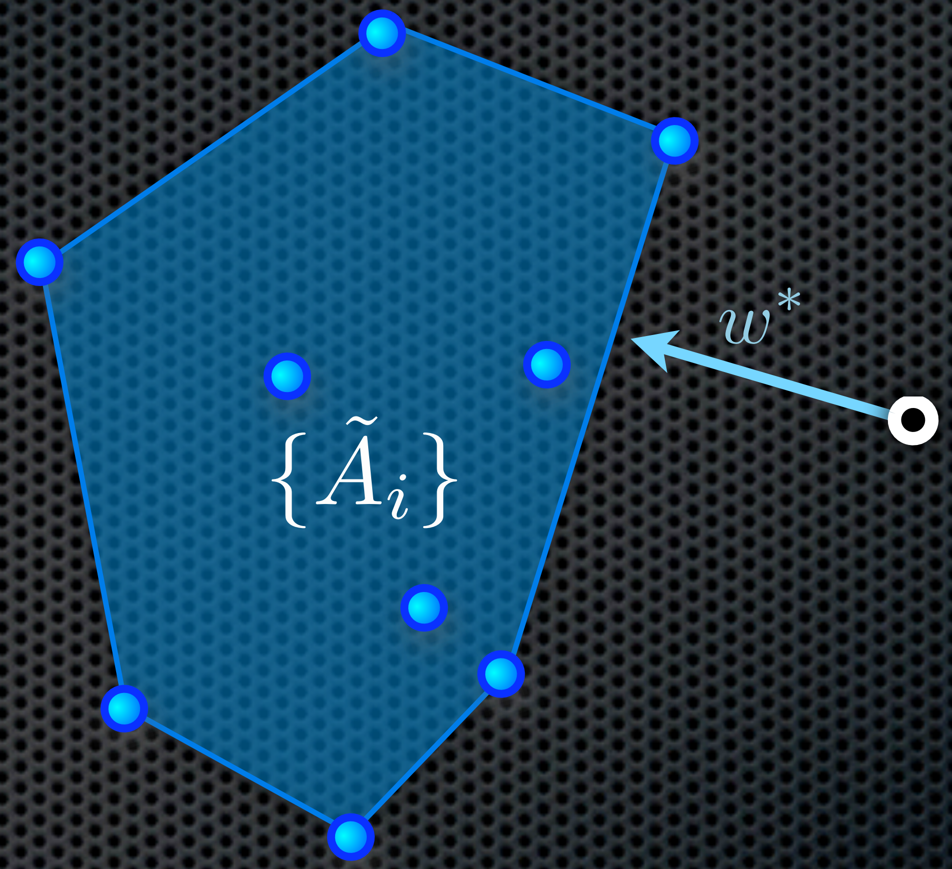


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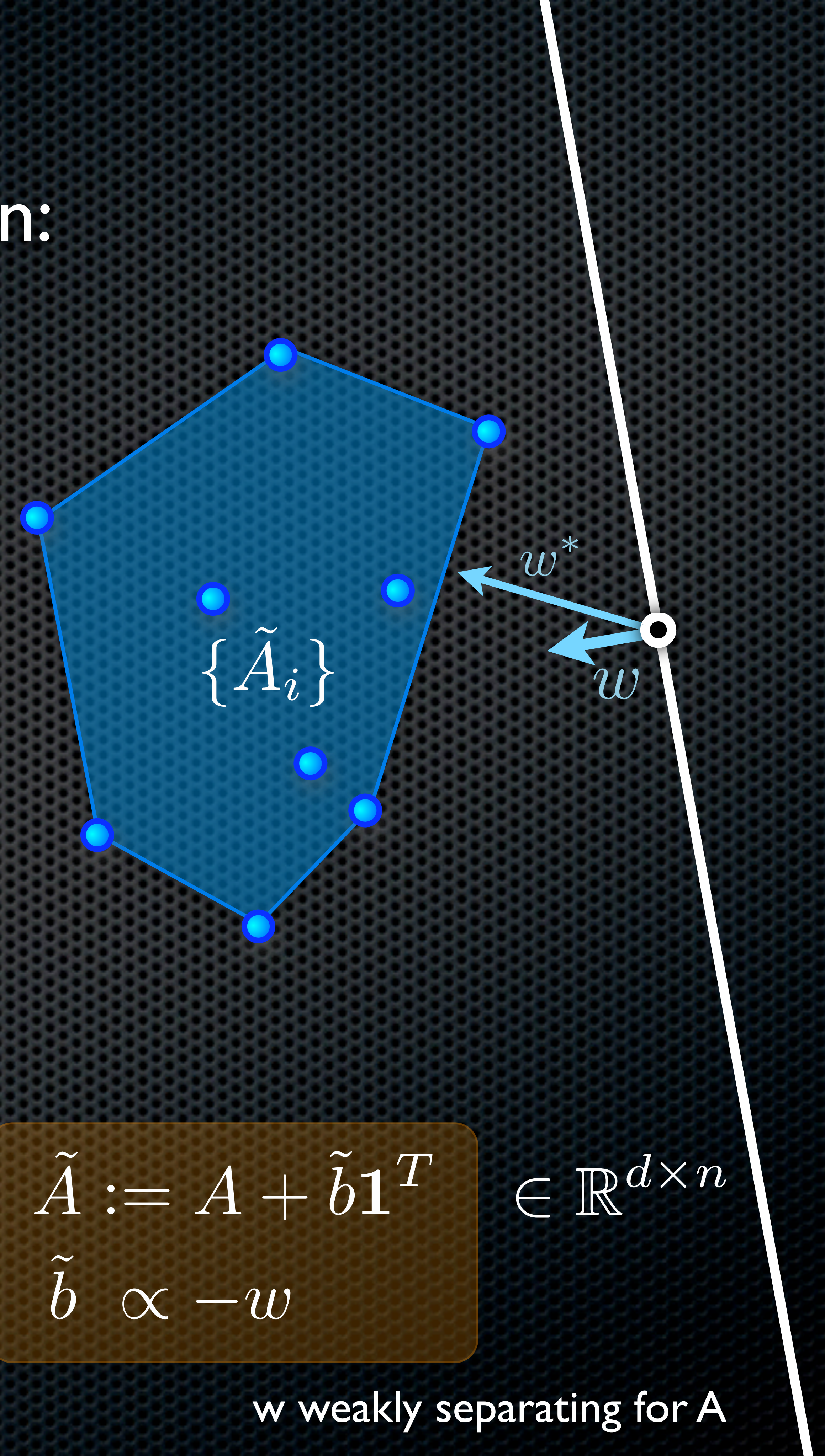


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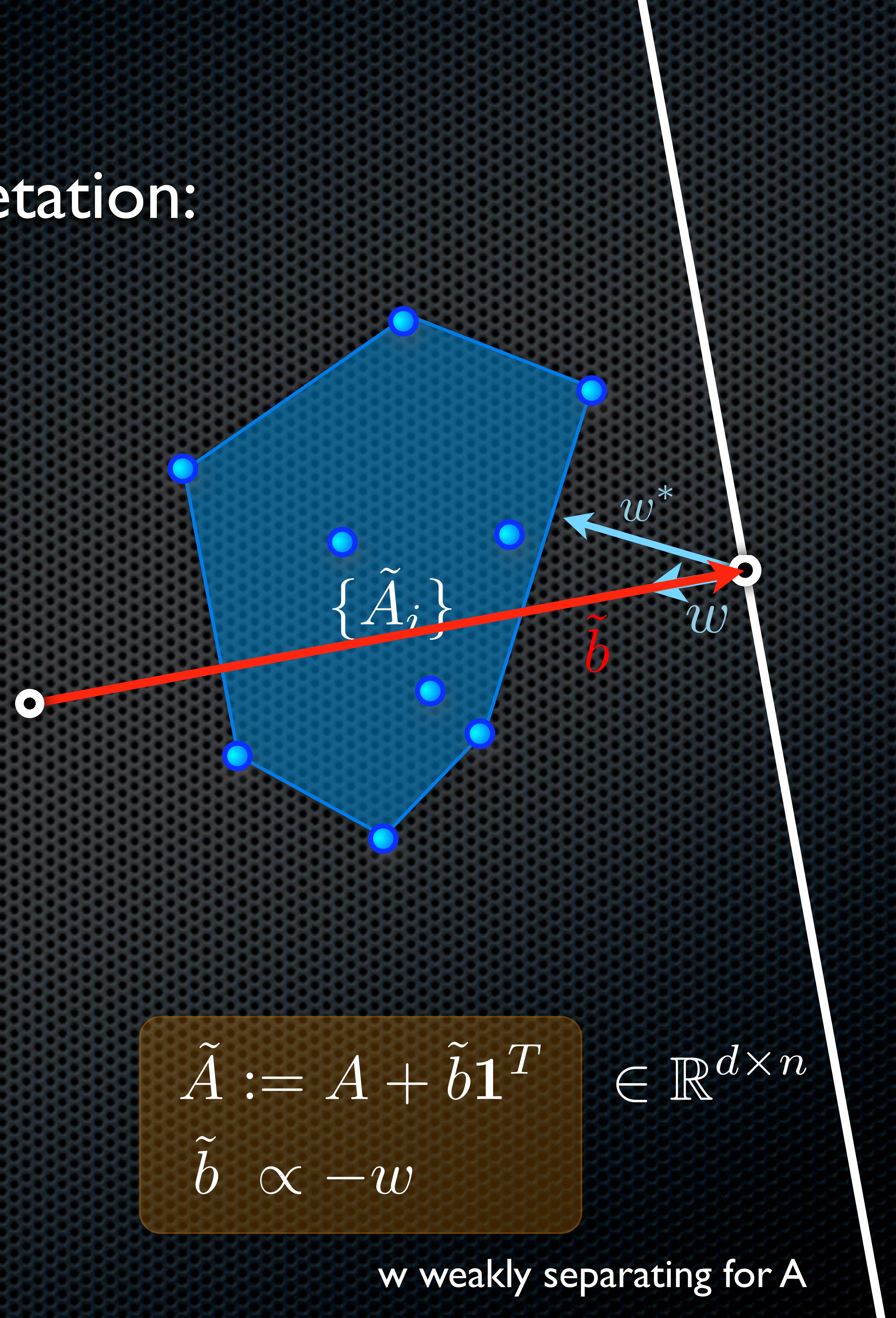


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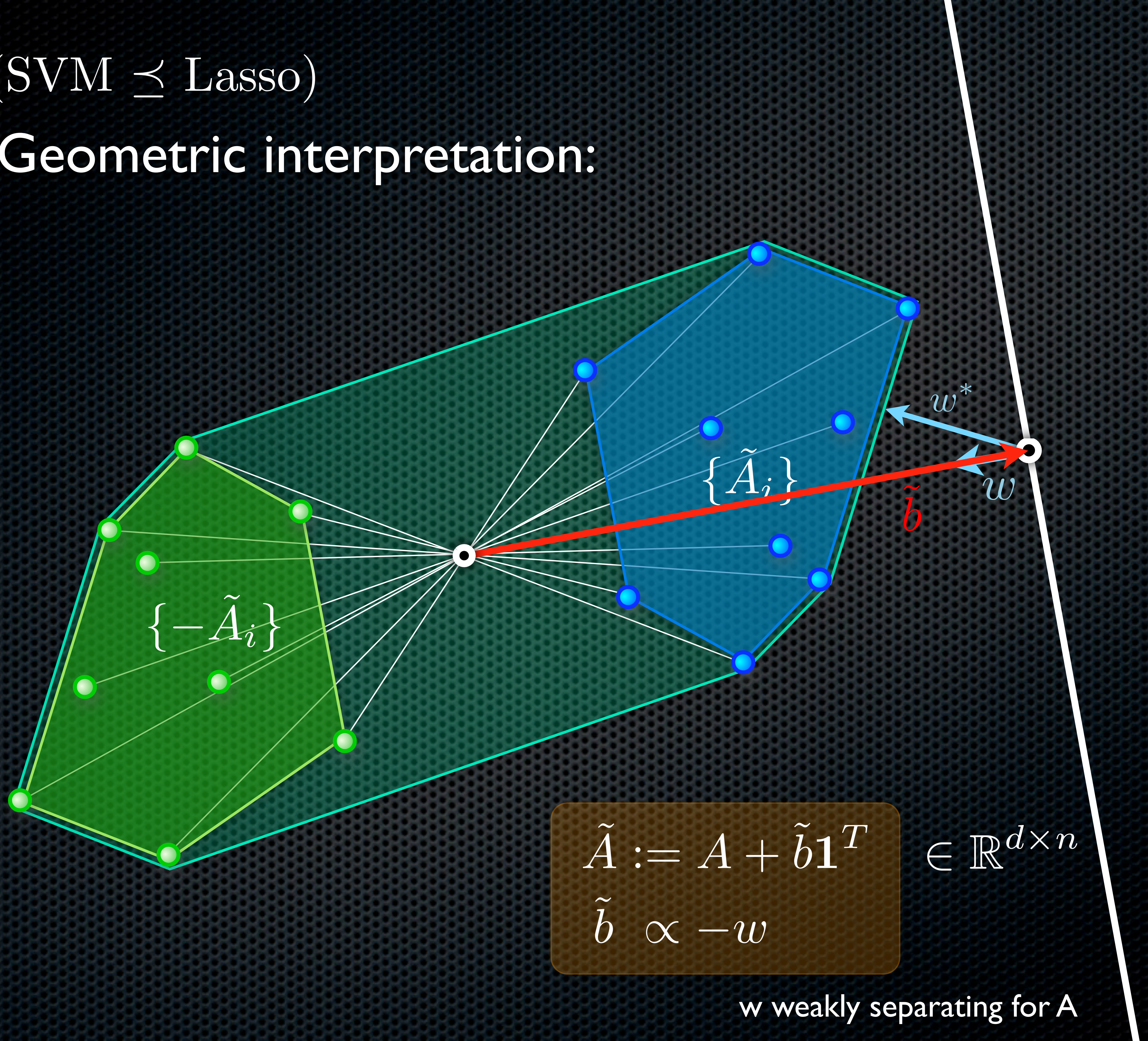


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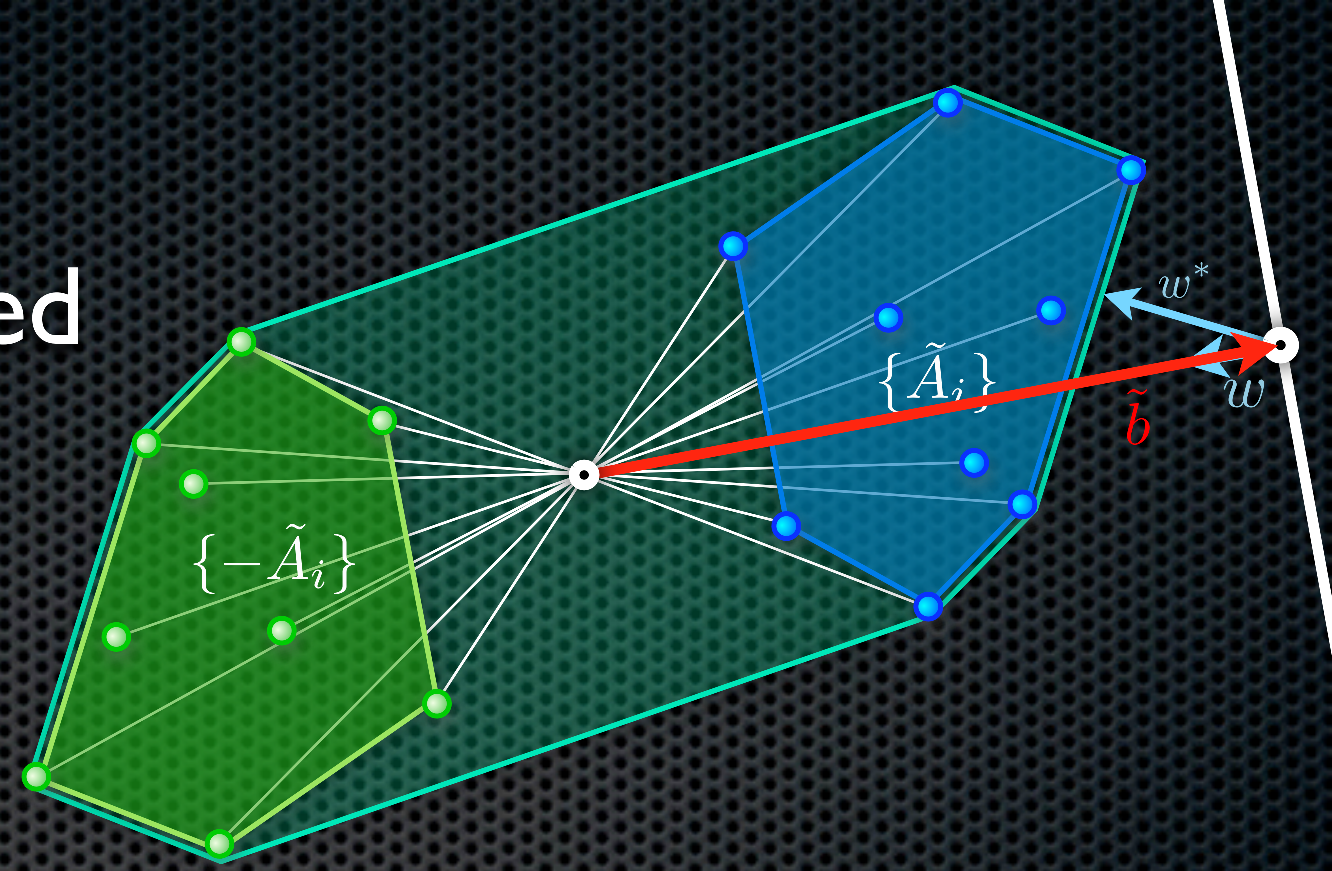
# Properties of the constructed Lasso instance

$$\min_{x \in L_1} \|\tilde{A}x - \tilde{b}\|^2$$




***Theorem:***

For any $x \in L_1$ for the Lasso, there is a vector $x' \in \Delta$, of the same or better Lasso objective.

This $x' \in \Delta$ attains the same objective in the SVM.

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- Kernelized version

$$\min_{x \in L_1} \left\| \sum_i \Psi(A_i) x_i - \Psi(b) \right\|_{\mathcal{H}}^2$$

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- Kernelized version

$$\min_{x \in L_1} \left\| \sum_i \Psi(A_i) x_i - \Psi(b) \right\|_{\mathcal{H}}^2$$

defined in terms of $\kappa(A_i, A_j),\ \kappa(A_i, b),\ \kappa(b, b)$ $\kappa(y,z) = \langle \Psi(y), \Psi(z) \rangle$

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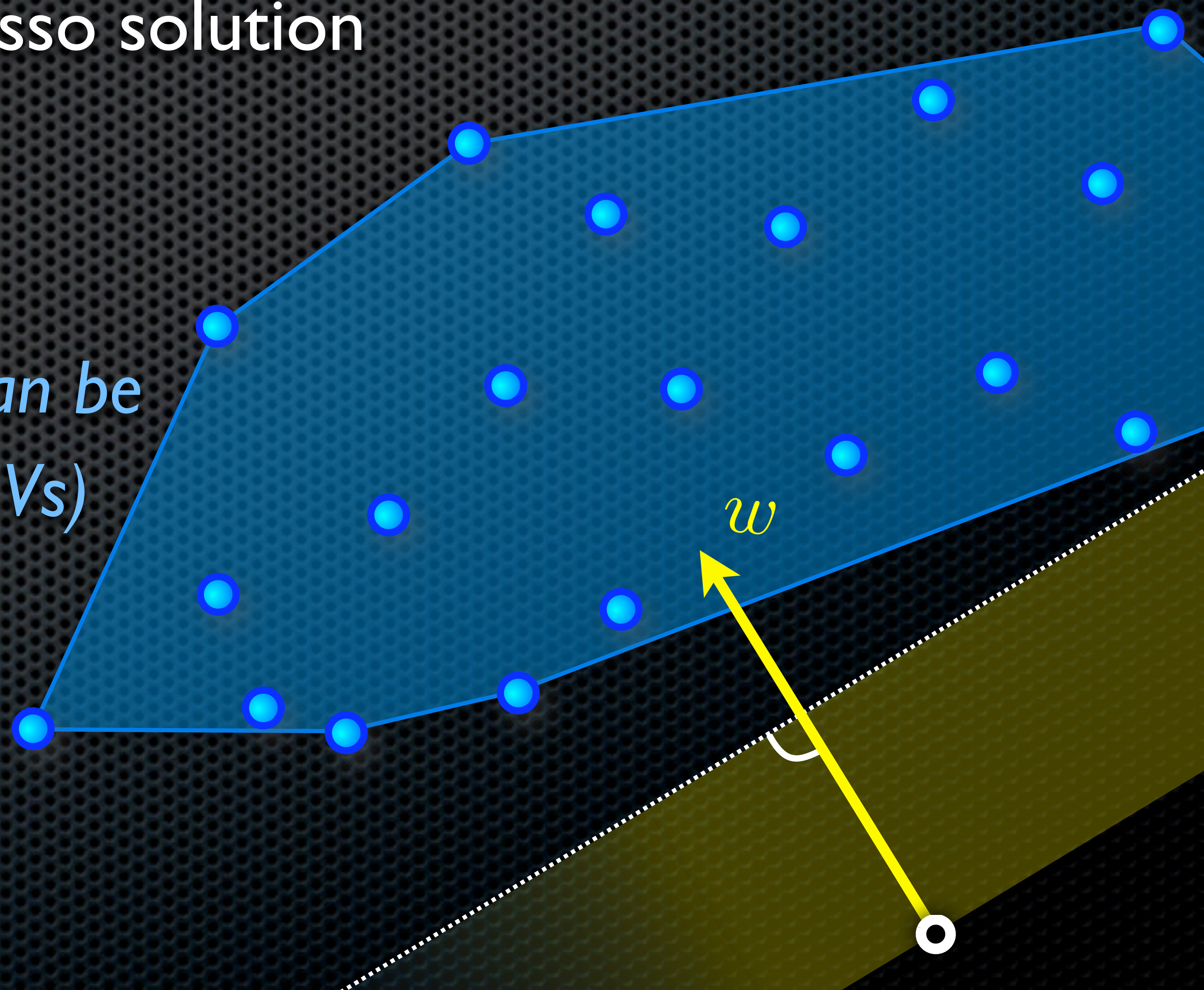

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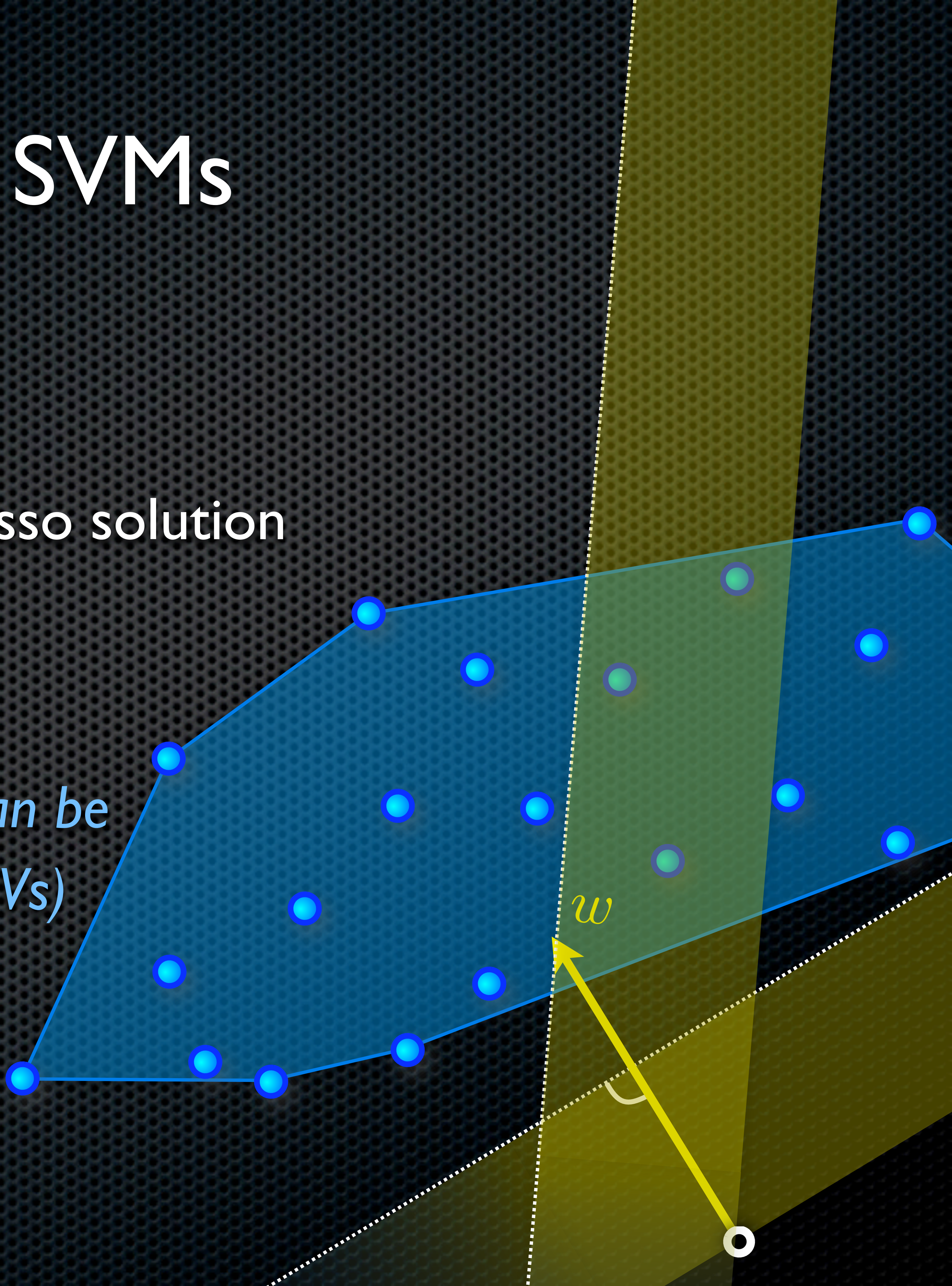

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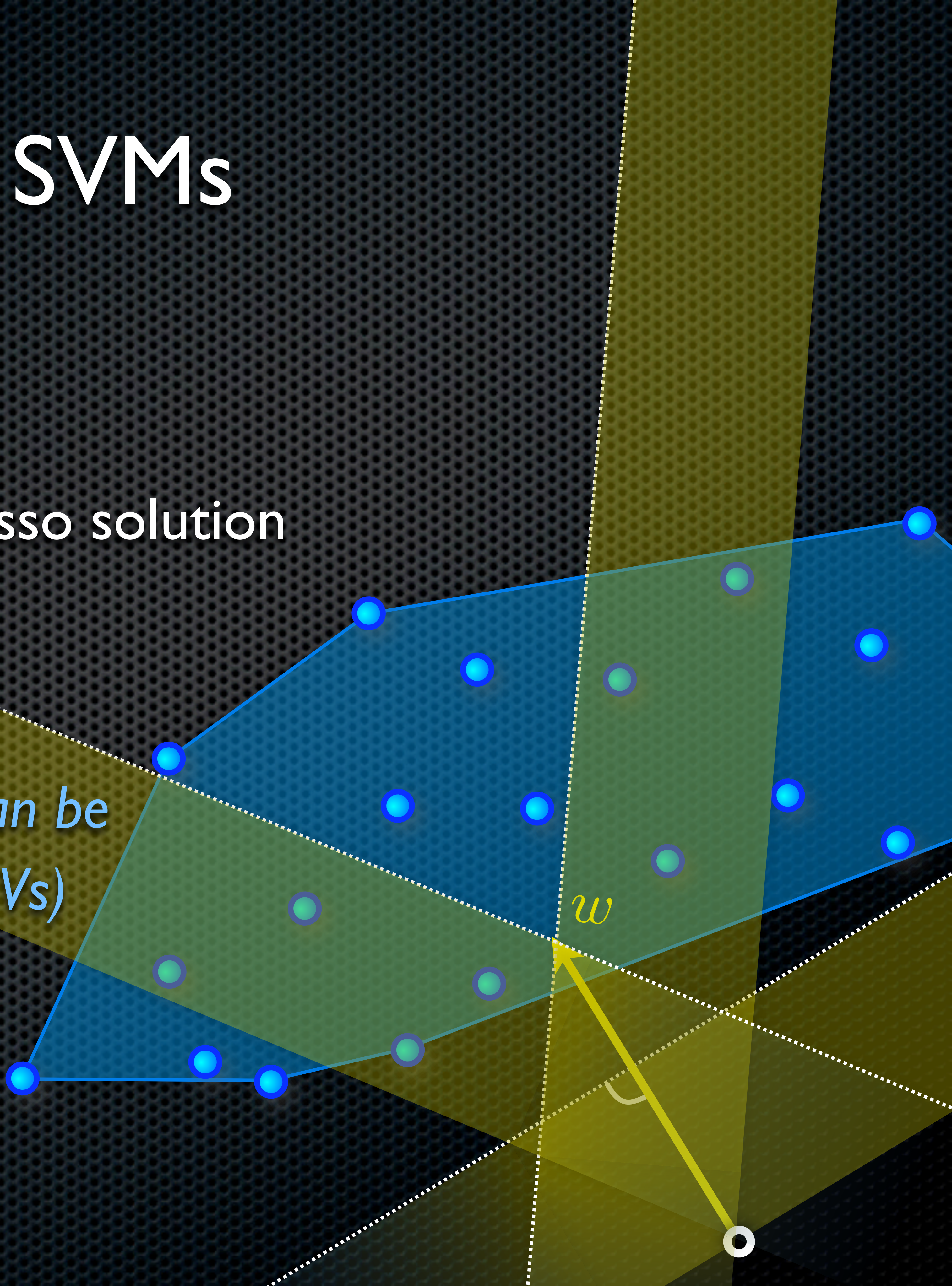

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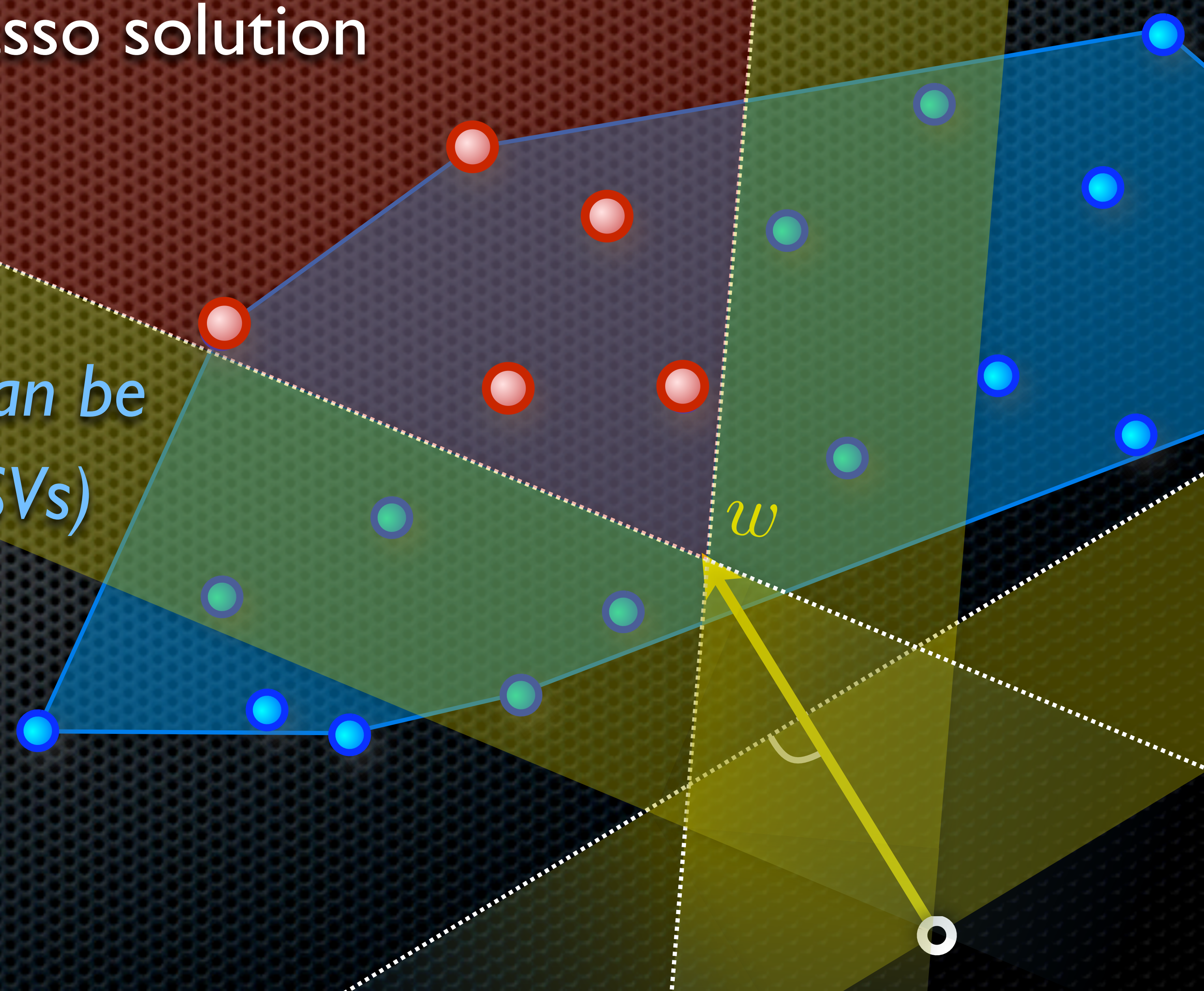

# Greedy Algorithms

## Convex optimization

methods applied to

$$\min_{x \in L_1} \|Ax - b\|^2$$

## Signal processing

sparse recovery methods

recover a sparse *x* from a noisy measurement *b* of *Ax*

# Greedy Algorithms

## Convex optimization

methods applied to

$$\min_{x \in L_1} \|Ax - b\|^2$$

Frank-Wolfe

## Signal processing

sparse recovery methods

recover a sparse *x* from a noisy measurement *b* of *Ax*

# Greedy Algorithms

## Convex optimization

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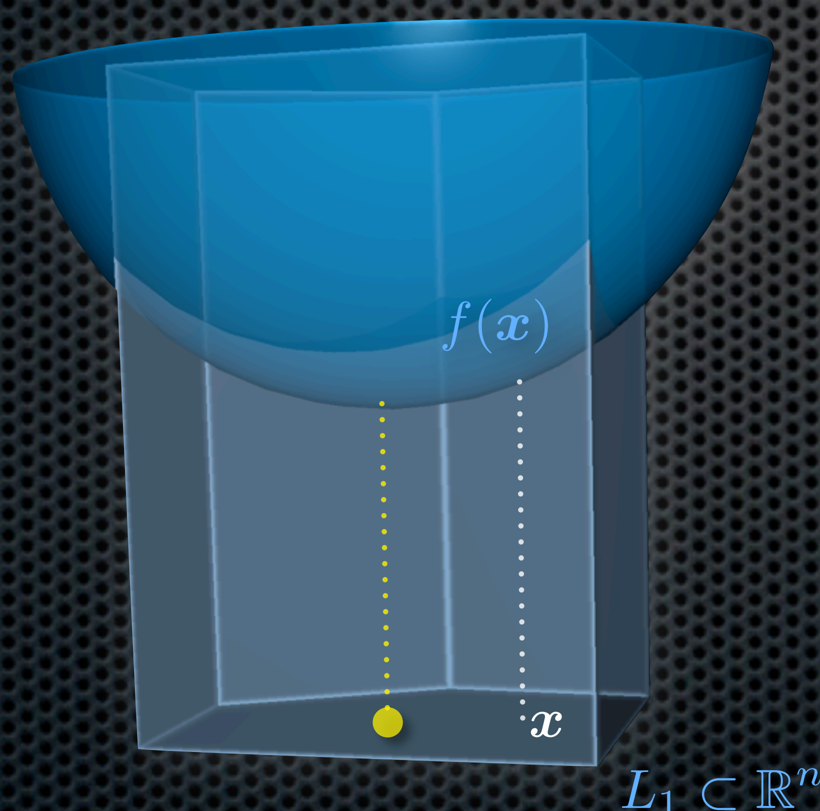


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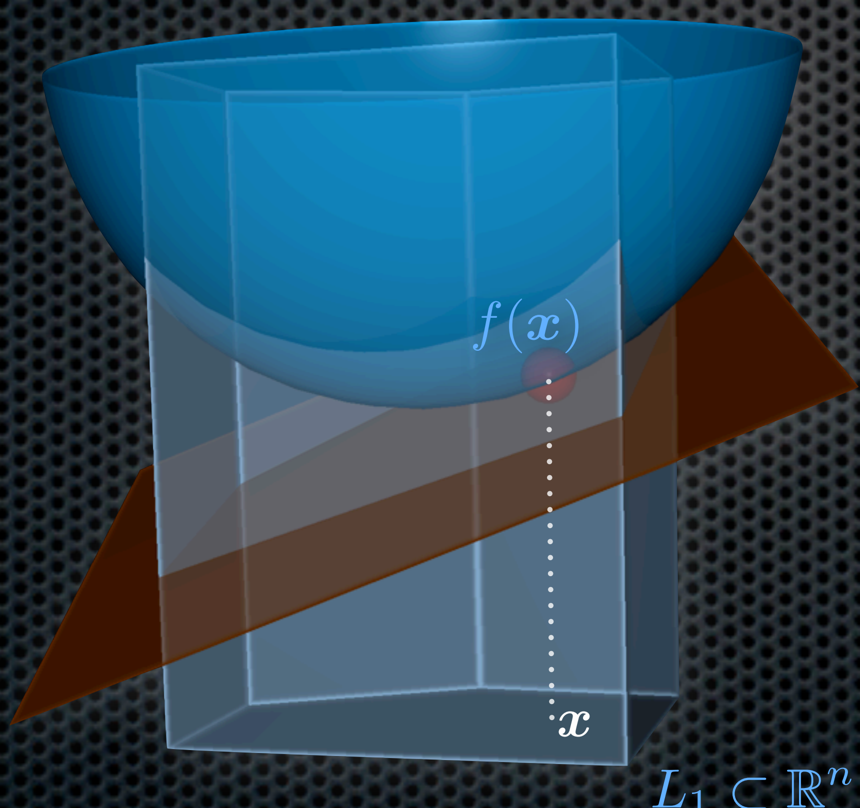


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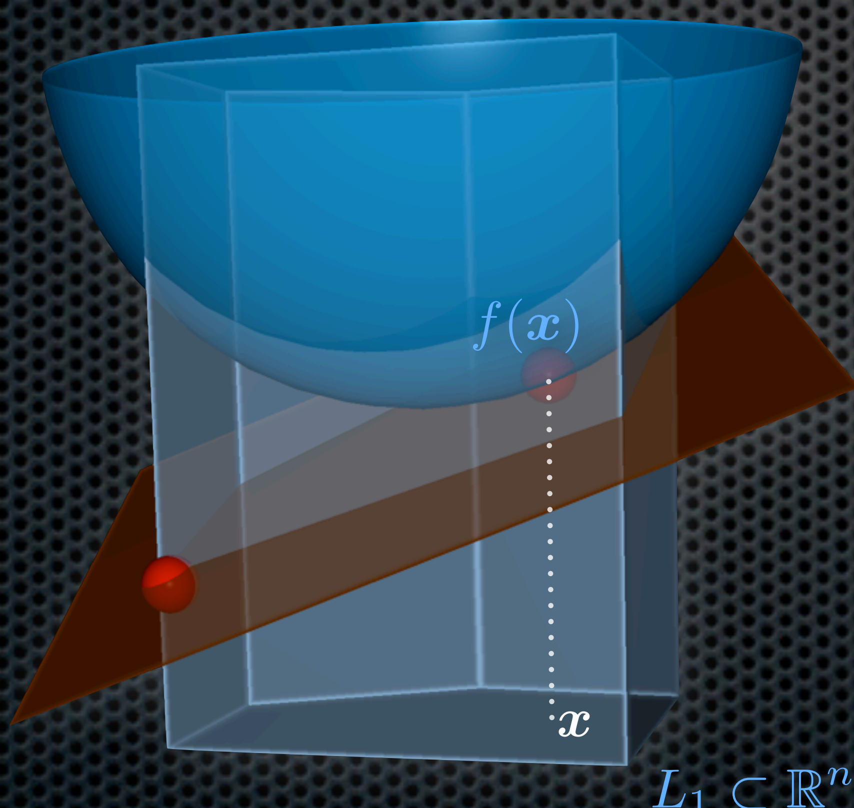


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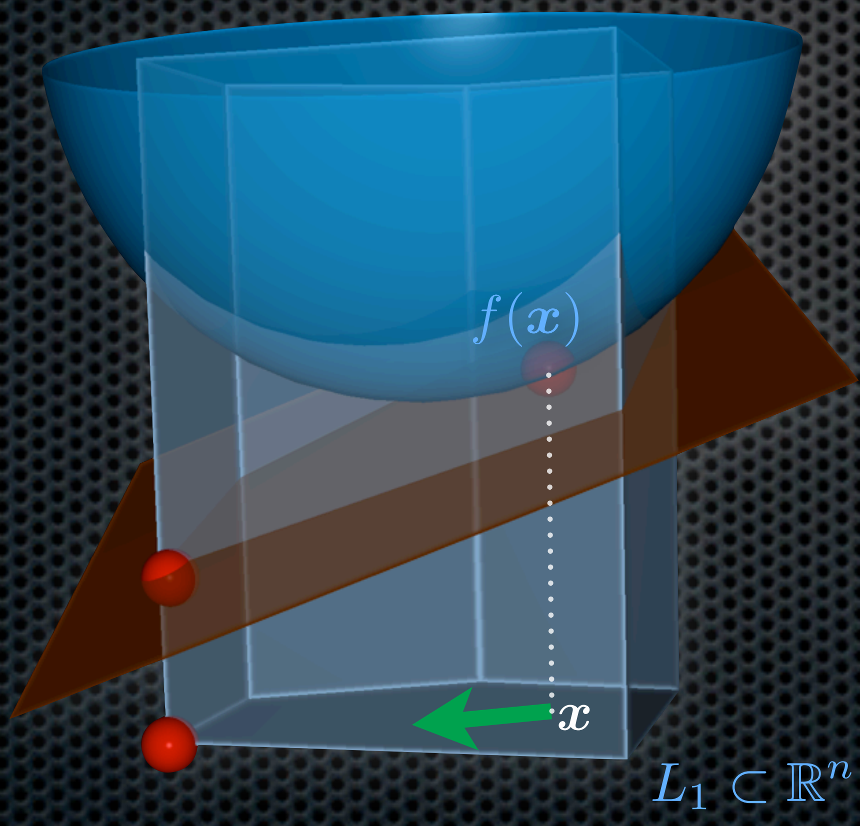


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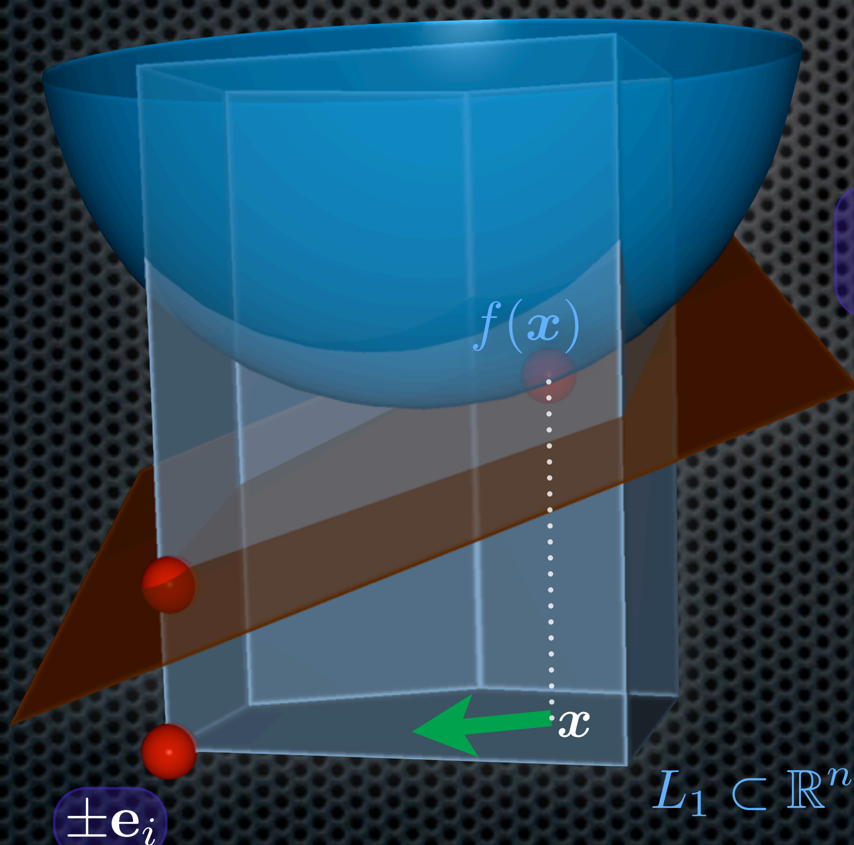


$$i := \arg\max_i |\nabla f(x)_i|$$

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Frank-Wolfe $\xleftrightarrow[\textit{atom per step}]{\textit{selects the same}}$ matching pursuit

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Frank-Wolfe ⟷ matching pursuit

*selects the same atom per step*

$$i := \arg\max_i |\nabla f(x)_i|$$

*fully corrective* Frank-Wolfe ⟷ OMP

*equivalent to*

Thanks