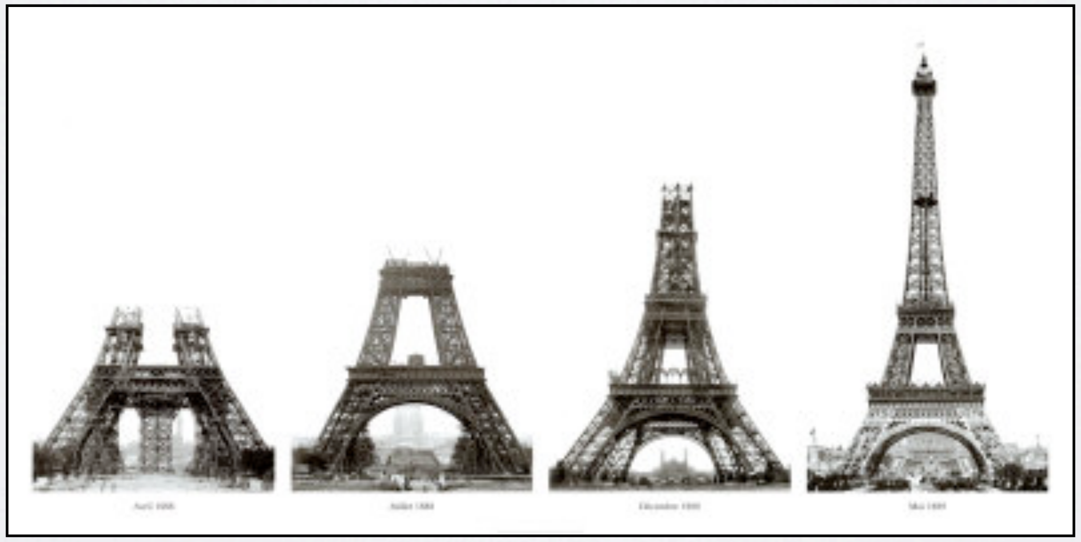
RANDOM MATRIX THEORY AND PRACTICE: OLD TRICKS FOR NEW DOGS

Pierpaolo Vivo
(LPTMS - CNRS - Paris XI)



Why are random matrix eigenvalues cool?

Message

- Ingredient: Take Any important mathematics
- Then Randomize!
- This will have many applications!

from a talk by Alan Edelman (MIT)

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

E. Artin (*Geometric Algebra*, p. 14)

John Wishart

Eugene Wigner

Freeman Dyson

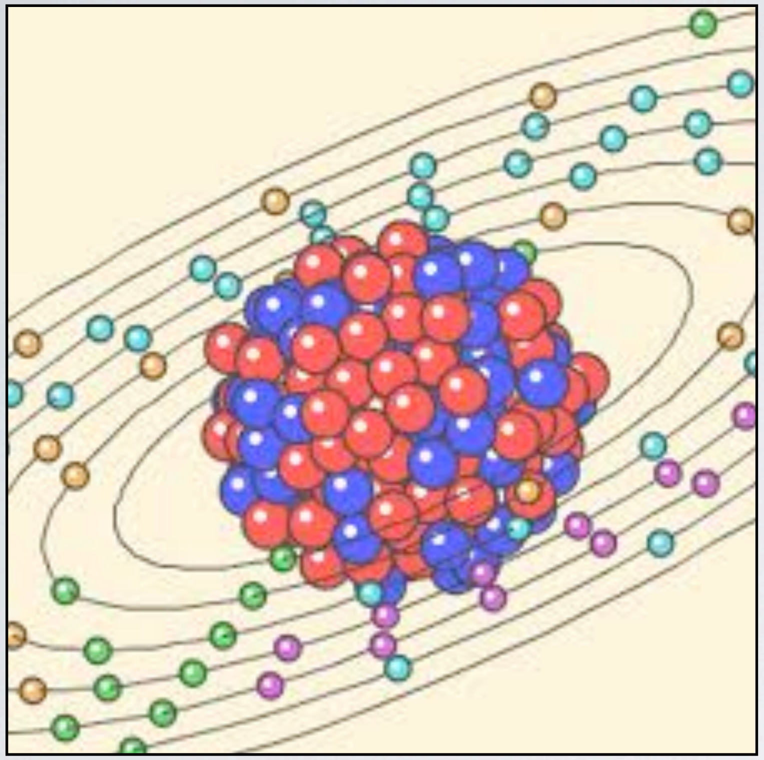
ANNALS OF MATHEMATICS
Vol. 67, No. 2, March, 1958
Printed in Japan

ON THE DISTRIBUTION OF THE ROOTS OF CERTAIN SYMMETRIC MATRICES

BY EUGENE P. WIGNER

(Received September 19, 1957)

The usual old story...



Hamiltonian (total energy) of heavy nuclei: hopeless task!

BUT.....

The Hamiltonian in a given basis is just a **HUGE** matrix....

Idea: take the matrix entries **at random...**

Random Matrices in Statistics

- Covariance estimation for the multivariate normal distribution

John Wishart

3. *Multi-variate Distribution. Use of Quadratic co-ordinates.*

A comparison of equation (8) with the corresponding results (1) and (2) for uni-variate and bi-variate sampling, respectively, indicates the form the general result may be expected to take. In fact, we have for the simultaneous distribution in random samples of the n variances (squared standard deviations) and the $\frac{n(n-1)}{2}$ product moment coefficients the following expression:

$$dp = \frac{\begin{vmatrix} A_{11} & A_{12} \dots & A_{1n} \\ A_{21} & A_{22} \dots & A_{2n} \\ \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} \dots & A_{nn} \end{vmatrix}^{\frac{N-1}{2}}}{(\sqrt{\pi})^{\frac{1}{2}n(n-1)}\,\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)\dots\Gamma\left(\frac{N-n}{2}\right)}$$

$$\times e^{-A_{11}a_{11}-A_{22}a_{22}-\dots-A_{nn}a_{nn}-2A_{12}a_{12}-2A_{13}a_{13}-\dots-2A_{n-1\,n}a_{n-1\,n}}$$

$$\times \begin{vmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots & a_{nn} \end{vmatrix}^{\frac{N-n-2}{2}} da_{11}da_{12}\dots da_{nn} \quad \dots\dots\dots\dots(9),$$

where $a_{pq} = s_p s_q r_{pq}$, and $A_{pq} = \frac{N}{2\sigma_p\sigma_q}\cdot\frac{\Delta_{pq}}{\Delta}$, Δ being the determinant

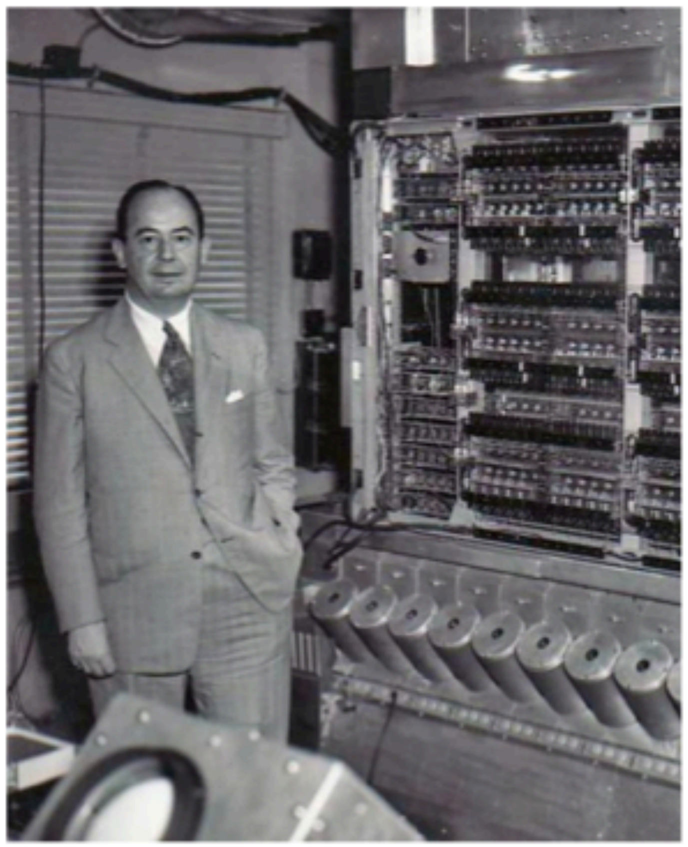
$$|\rho_{pq}|,\ p, q = 1, 2, 3, \dots n,$$

and Δ_{pq} the minor of ρ_{pq} in Δ.

[Refs] Wishart, *Biometrika* 1928. Photo from apprendre-math.info.

Random Matrices in Numerical Linear Algebra

- Model for floating-point errors in LU decomposition



John von Neumann

now combining (8.6) and (8.7) we obtain our desired result:

$$\text{Prob}\,(\lambda > 2\sigma^2 rn) < \frac{(rn)^{n-1/2}e^{-rn}\pi^{1/2}e^n\cdot 2^{n-2}}{\pi n^{n-1}(r-1)n} \tag{8.8}$$

$$= \left(\frac{2r}{e^{r-1}}\right)^n \times \frac{1}{4(r-1)(r\pi n)^{1/2}}.$$

We sum up in the following theorem:

(8.9) The probability that the upper bound $|A|$ of the matrix A of (8.1) exceeds $2.72\sigma n^{1/2}$ is less than $.027 \times 2^{-n} n^{-1/2}$, that is, with probability greater than 99% the upper bound of A is less than $2.72\sigma n^{1/2}$ for $n = 2, 3, \cdots$.

This follows at once by taking $r = 3.70$.

[Refs] von Neumann and Goldstine, *Bull. AMS* 1947 and *Proc. AMS* 1951. Photo ©IAS Archive.

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These articles have been made available online, by the Annals of Human Genetics, UCL and Blackwell Publishing Ltd strictly for historical and academic reasons. The work of eugenicists was often pervaded by prejudice against racial, ethnic and disabled groups. Publication of this material online is for scholarly research purposes is not an endorsement or promotion of the views expressed in any of these articles or eugenics in general. All articles are published in full, except where necessary to protect individual privacy.
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the simultaneous distribution of the p variates ϕ is seen to be

$$\frac{\pi^{\frac{1}{2}p}}{(\frac{1}{2}n-1)!\dots(\frac{1}{2}n-\frac{1}{2}p-\frac{1}{2})!\,(\frac{1}{2}p-1)!\dots(-\frac{1}{2})!}$$

$$\times e^{-\phi_1-\dots-\phi_p}(\phi_1\dots\phi_p)^{\frac{1}{2}n-\frac{1}{2}p-\frac{1}{2}}(\phi_1-\phi_2)\dots(\phi_{p-1}-\phi_n)\,d\phi_1\dots d\phi_p,$$

where $$0<\phi_p<\phi_{p-1}<\dots<\phi_1<\infty. \qquad \dots\dots(11)$$

[R.A. Fisher, 1939]

N = 5

$$\begin{pmatrix} 0.5377 & 0.2631 & -1.8044 & 0.3286 & 0.4951 \\ 0.2631 & -0.4336 & 1.6888 & 1.7271 & 0.7810 \\ -1.8044 & 1.6888 & 0.7254 & 0.7133 & 0.7160 \\ 0.3286 & 1.7271 & 0.7133 & 1.4090 & 1.5237 \\ 0.4951 & 0.7810 & 0.7160 & 1.5237 & 0.4889 \end{pmatrix}$$

$$\vec{\lambda} = \begin{bmatrix} -2.4341 & -0.8386 & -0.5203 & 2.2594 & 4.2610 \end{bmatrix}$$

Typically we are interested in $N \to \infty$, but sometimes...

Basic Goal of RMT

From $\mathcal{P}(H_{11}, \ldots, H_{NN})$ Joint Probability Density of Entries

To ... as much as we can about the eigenvalues

- Average density
- Spacings
- Largest and smallest
- ......

Ideally....

$$\mathcal{P}(H_{11}, \ldots, H_{NN})$$

↓

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N)$$

Not always possible!

"When a distinguished but elderly scientist states that something is possible, he is almost certainly right. When he states that something is impossible, he is very probably wrong." (Arthur C. Clarke)

$N = 5$

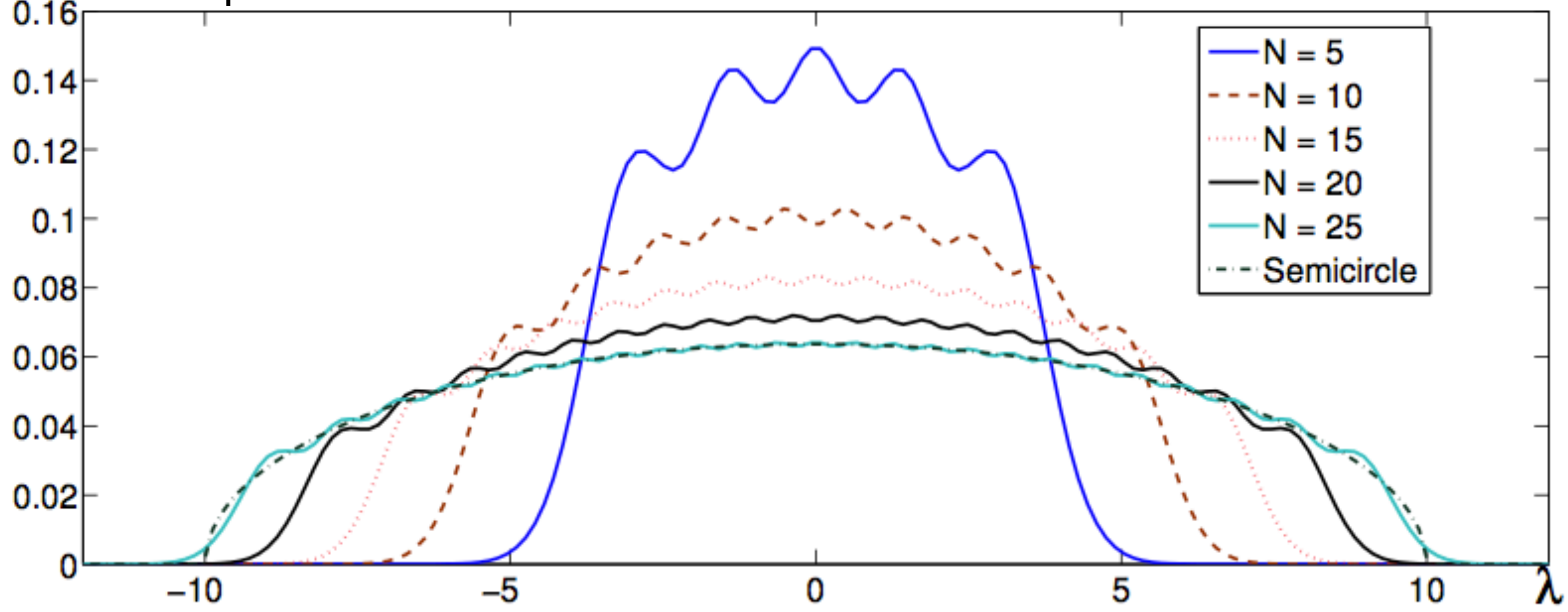
$$\begin{pmatrix} 0.5377 & 0.2631 & -1.8044 & 0.3286 & 0.4951 \\ 0.2631 & -0.4336 & 1.6888 & 1.7271 & 0.7810 \\ -1.8044 & 1.6888 & 0.7254 & 0.7133 & 0.7160 \\ 0.3286 & 1.7271 & 0.7133 & 1.4090 & 1.5237 \\ 0.4951 & 0.7810 & 0.7160 & 1.5237 & 0.4889 \end{pmatrix}$$

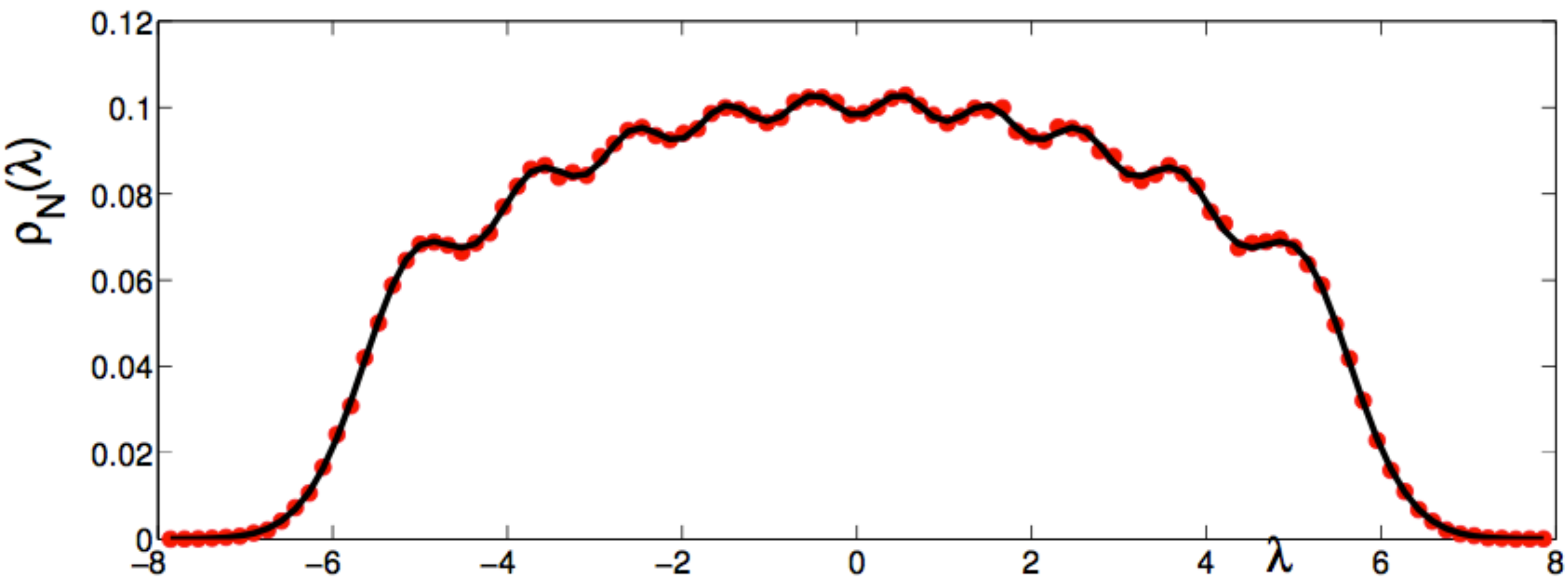
$$\vec{\lambda} = \begin{bmatrix} -2.4341 & -0.8386 & -0.5203 & 2.2594 & 4.2610 \end{bmatrix}$$

Let's repeat the experiment many times and histogram all the eigenvalues...

"Average Spectral Density"

p. 27

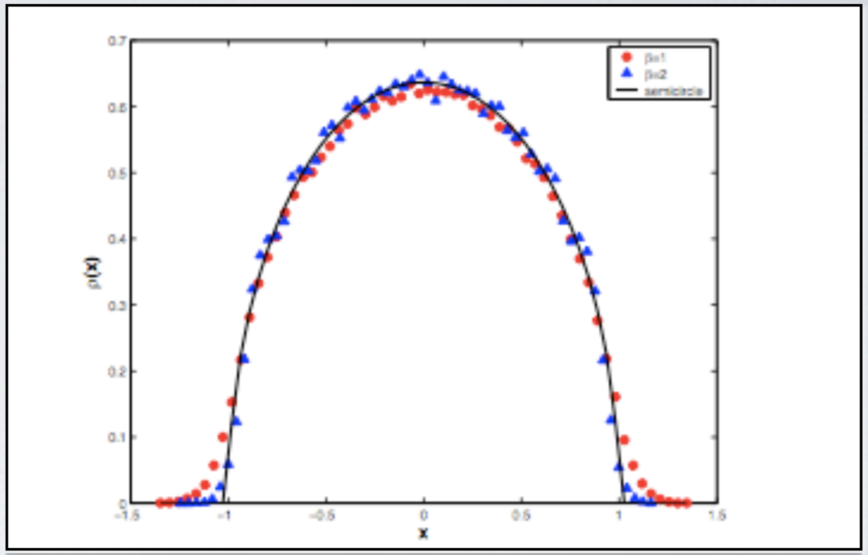

N = 5
N = 10
N = 15
N = 20
N = 25
Semicircle
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
−10
−5
0
5
10
λ


ρN(λ)
0.12
0.1
0.08
0.06
0.04
0.02
0
−8
−6
−4
−2
0
2
4
6
8
λ

Wigner's "Semicircle" Law

$$\rho_{N\to\infty}(\lambda) \to \frac{1}{\sqrt{2\beta N}} f\left(\frac{\lambda}{\sqrt{2\beta N}}\right)$$

$$f(x) = \frac{2}{\pi}\sqrt{1-x^2}$$



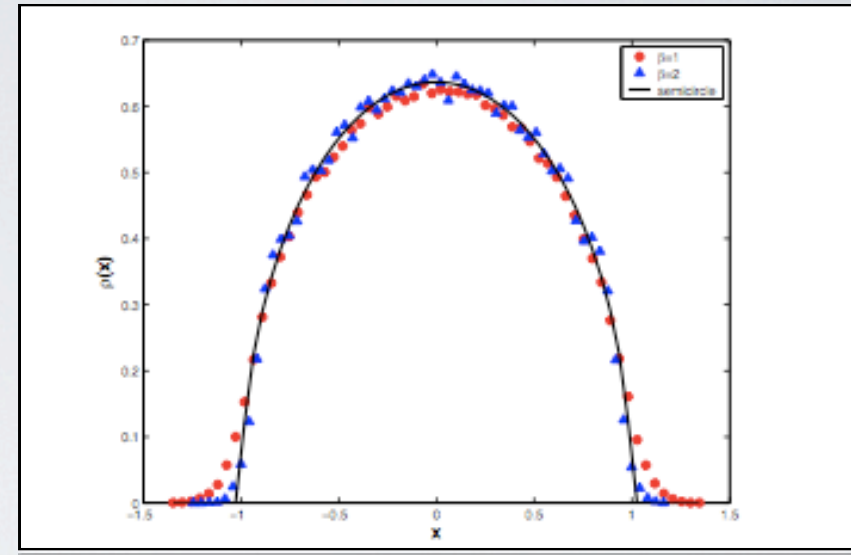
$$\beta = 1, 2, 4$$

Dyson's "threefold way"

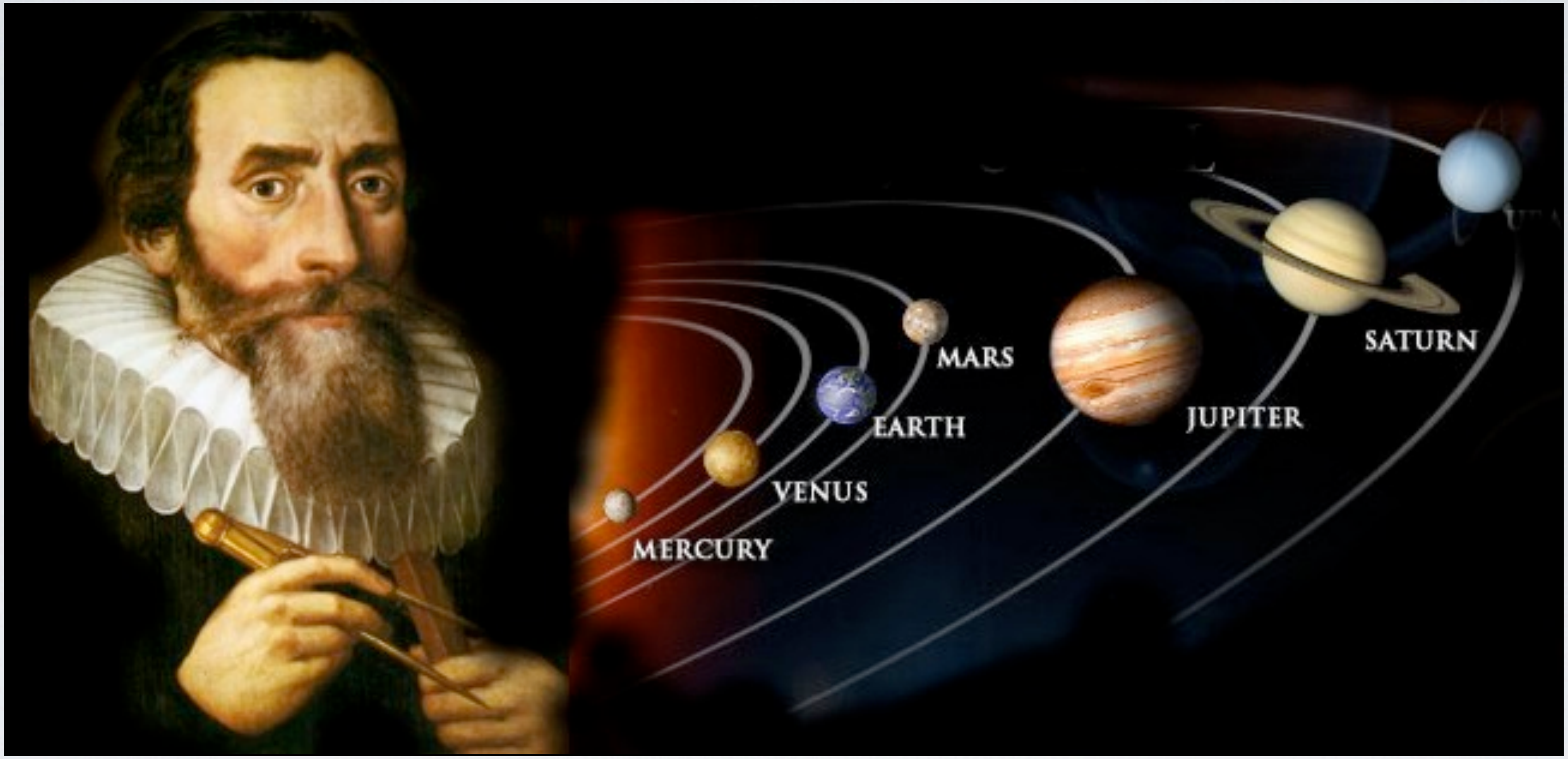
Wigner's "Semicircle" Law

$$\rho_{N\to\infty}(\lambda) \to \frac{1}{\sqrt{2\beta N}} f\left(\frac{\lambda}{\sqrt{2\beta N}}\right)$$

$$f(x) = \frac{2}{\pi}\sqrt{1 - x^2}$$



...which btw is **not** a semicircle



Johannes Kepler (1571-1630)

ANNALS OF MATHEMATICS
Vol. 62, No. 3, November, 1955

CHARACTERISTIC VECTORS OF BORDERED MATRICES WITH INFINITE DIMENSIONS

BY EUGENE P. WIGNER

(Received April 18, 1955)

The statistical properties of th

Random sign symmetric matrix

The matrices to be considered are $2N + 1$ dimensional real symmetric matrices; N is a very large number. The diagonal elements of these matrices are zero, the non diagonal elements $v_{ik} = v_{ki} = \pm v$ have all the same absolute value but random signs. There are $\mathfrak{N} = 2^{N(2N+1)}$ such matrices. We shall calculate, after

Comparison of (19b) with (19) yields

$$\rho(\xi) = (4\pi q)^{-1}(8q - \xi^2)^{\frac{1}{2}} \quad \text{for } \xi^2 < 8q \tag{20}$$

$$= 0 \quad \text{elsewhere.}$$

First occurrence (?) of the "semicircle" law in RMT.
Originally **not** derived for Gaussian matrices!

Possible questions...

- Is semicircle law "universal" ?
- If not, can we derive the corresponding spectral density of **any** matrix model?
- If we can't ..... why??

Classification!

Matrices with **real** eigenvalues: a layman's classification

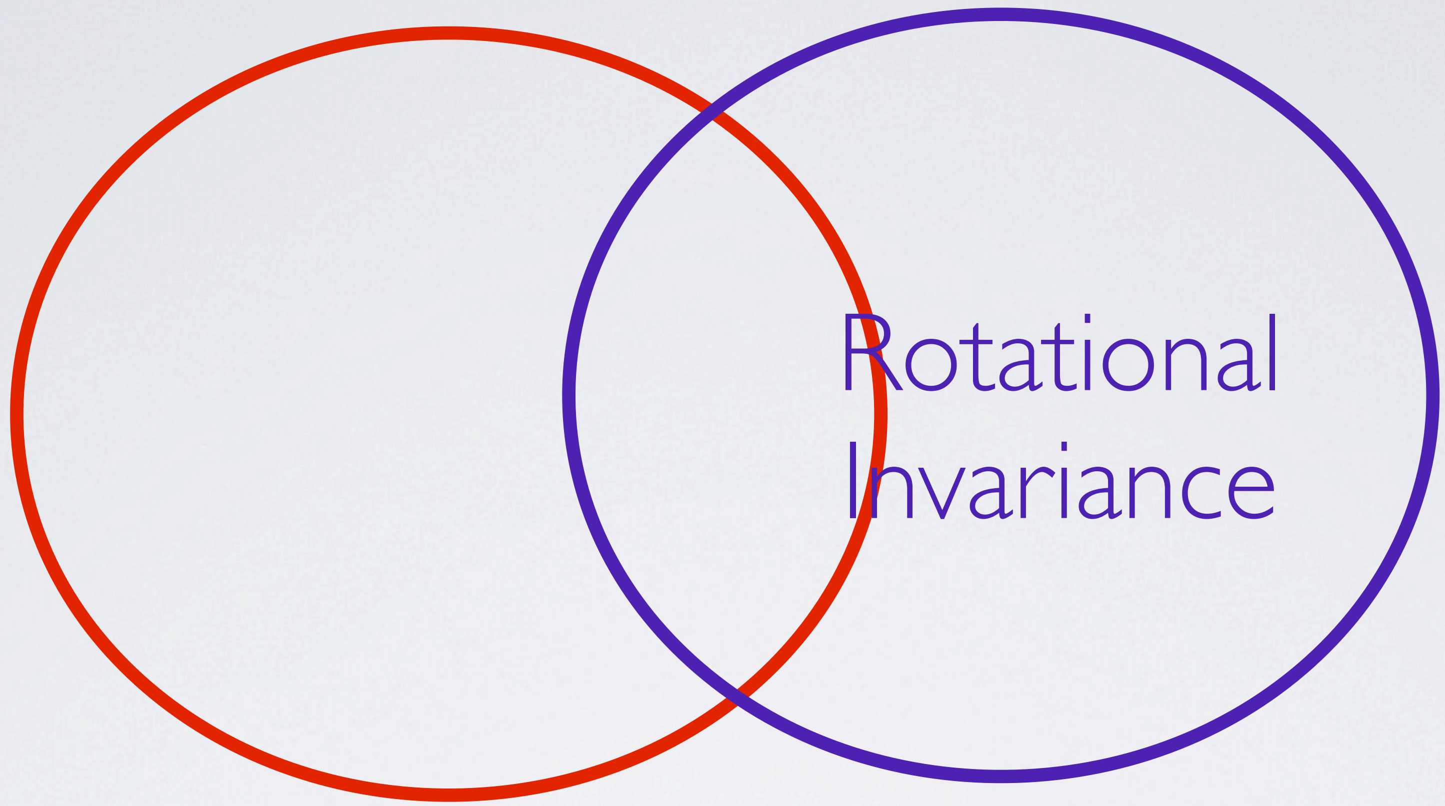
Independent
Entries

$$\mathcal{P}(\mathbf{H}) = \prod_{i=1}^{N} f_i(x_{ii}) \prod_{i<j}^{N} f_{ij}^{(1)}(x_{ij}) f_{ij}^{(2)}(y_{ij}),$$

...called Wigner matrices

careful!

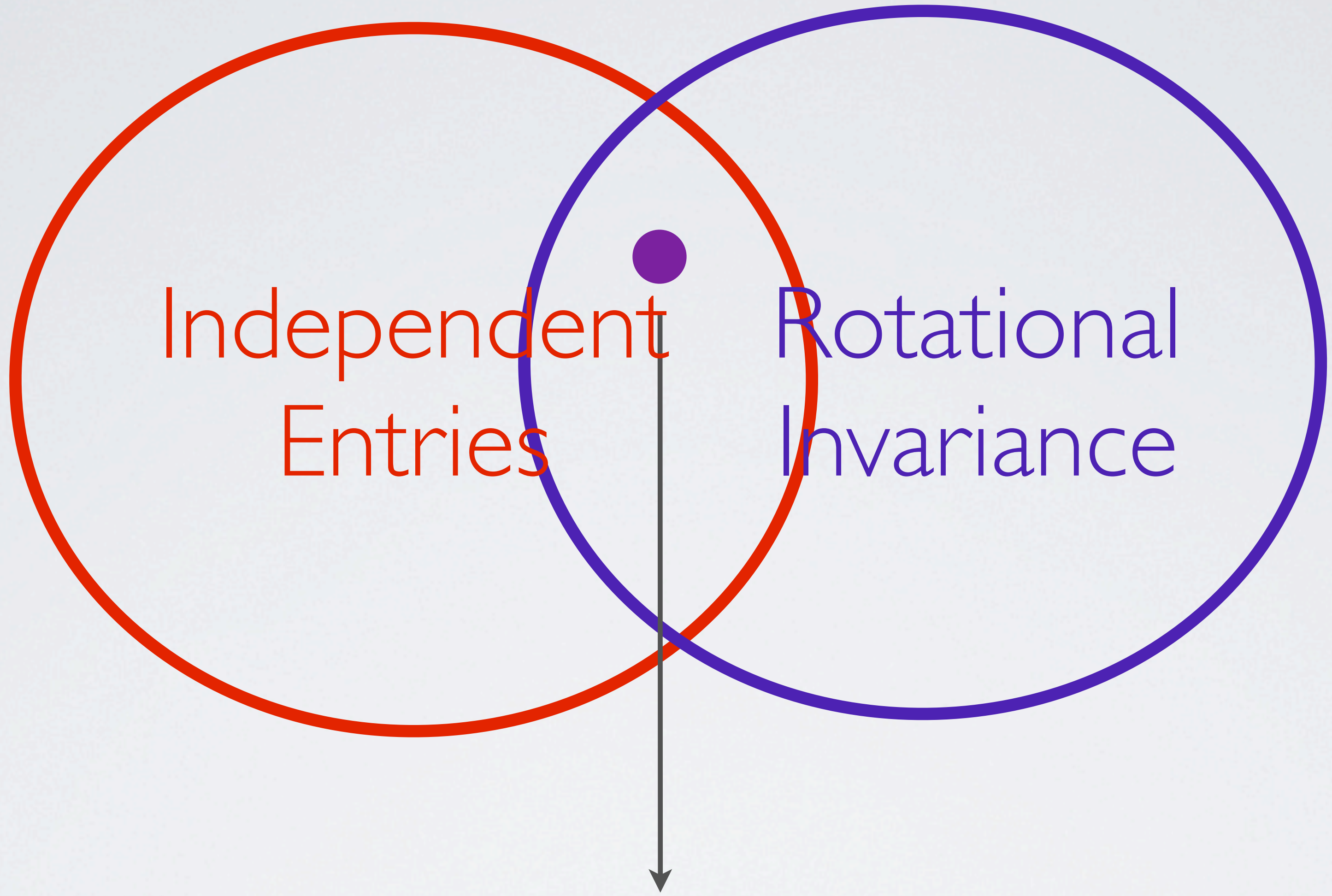
Matrices with **real** eigenvalues: a layman's classification



$$\mathcal{P}(\mathbf{H}) = \mathcal{P}(\mathbf{U}\mathbf{H}\mathbf{U}^{-1})$$

...this means that eigenvectors are not that important!

Matrices with **real** eigenvalues: a layman's classification



Gaussian Ensemble

$$\mathcal{P}(H_{11},\ldots,H_{NN}) \propto \prod_i e^{-H_{ii}^2/2\sigma^2} \prod_{j>k} e^{-2H_{jk}^2/2\sigma^2}$$

$$\propto e^{-\frac{1}{2\sigma^2}\mathrm{Tr}(\mathbf{H}^2)}$$

GOE

Porter-Rosenzweig Theorem (1960)

pag. 11

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ANNALES ACADEMIÆ SCIENTIARUM FENNICÆ

SARJA
SERIES A

VI. PHYSICA

44

STATISTICAL PROPERTIES OF ATOMIC AND NUCLEAR SPECTRA

BY

CHARLES E. PORTER

School of Physics, University of Minnesota,
Minneapolis, Minnesota

NORBERT ROSENZWEIG

Physics Division, Argonne National Laboratory,
Lemont, Illinois

The Gaussian ensemble

$$\mathcal{P}(H_{11},\ldots,H_{NN}) \propto \prod_i e^{-H_{ii}^2/2\sigma^2} \prod_{j>k} e^{-2H_{jk}^2/2\sigma^2}$$

$$\propto e^{-\frac{1}{2\sigma^2}\mathrm{Tr}(\mathbf{H}^2)}$$

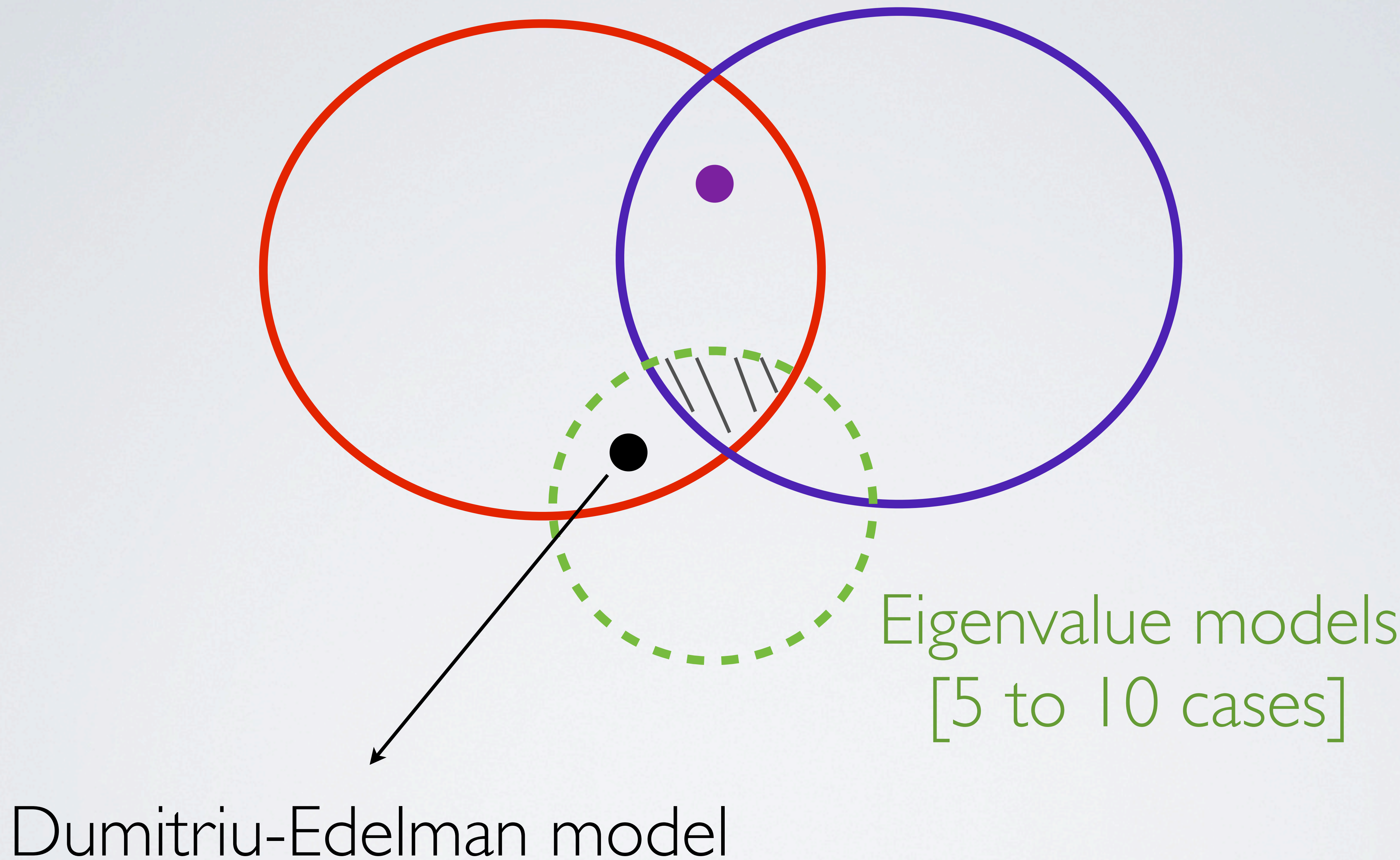
$$\mathcal{P}(\mathbf{H}) = \mathcal{P}(\mathbf{UHU}^{-1})$$

It was shown by Porter and Rosenzweig[4] that the special form of Eq. (1) is implied by two apparently more general requirements: (i) the various components H_{ij} to be statistically independent, and (ii) the function $D(H_{ij})$ to be invariant under all transformations

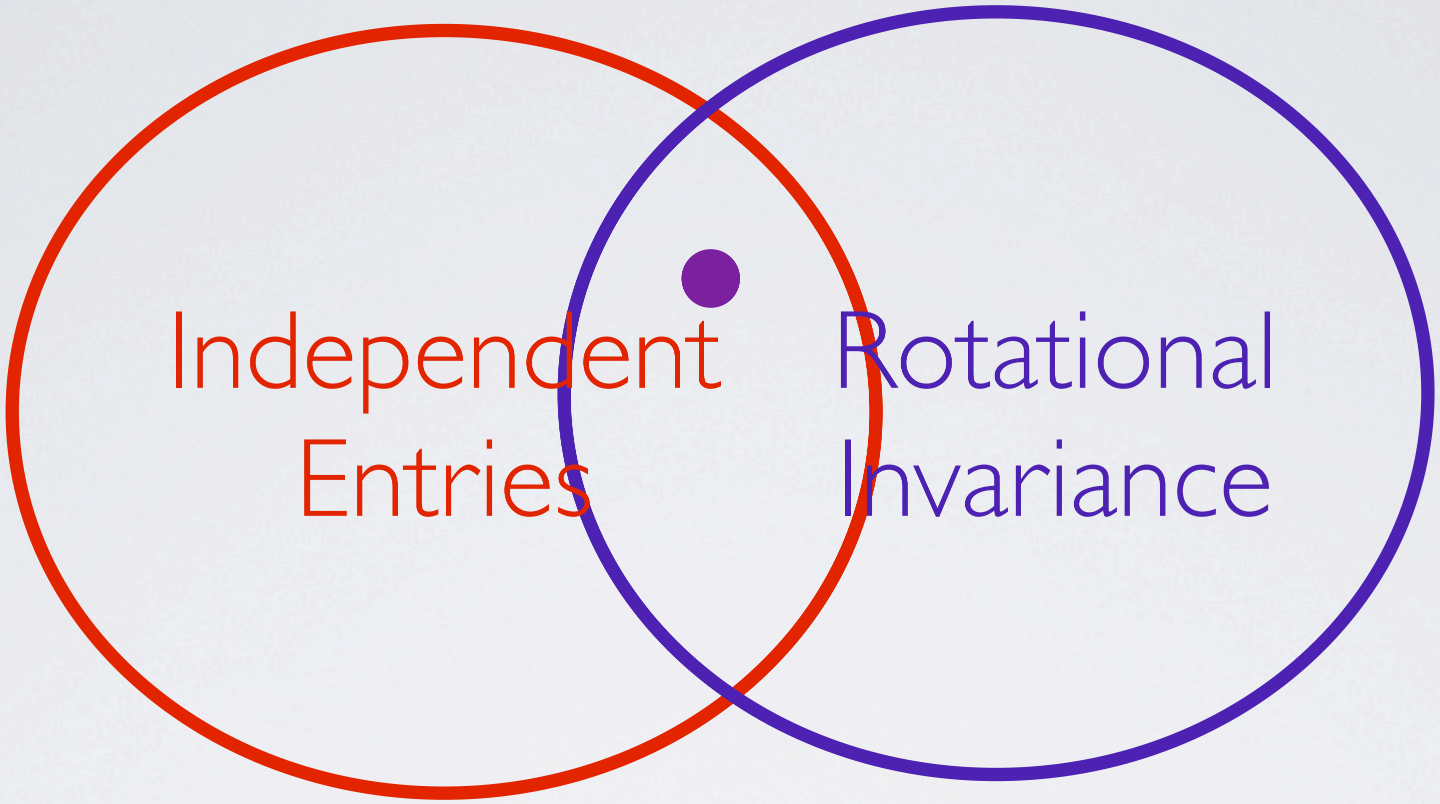
$H \rightarrow R^{-1}HR$, where R is a real orthogonal matrix. The requirement (ii) is a natural one in any ensemble that attempts to give equal weight to all kinds of interactions. However, requirement (i) is artificial and without clear physical motivation. To picture the H_{ij} as resulting from some "random process" of a conventional kind does not seem reasonable. Therefore the definition of E_G remains somewhat arbitrary.

The basic reason for the unsatisfactory features of Eq. (1) is that one cannot define a uniform probability distribution on an infinite range. Thus some arbitrary restriction of the magnitudes of the H_{ij} is inevitable. It is impossible to define an ensemble in terms of the H_{ij} in which all interactions are equally probable.

Anything else?

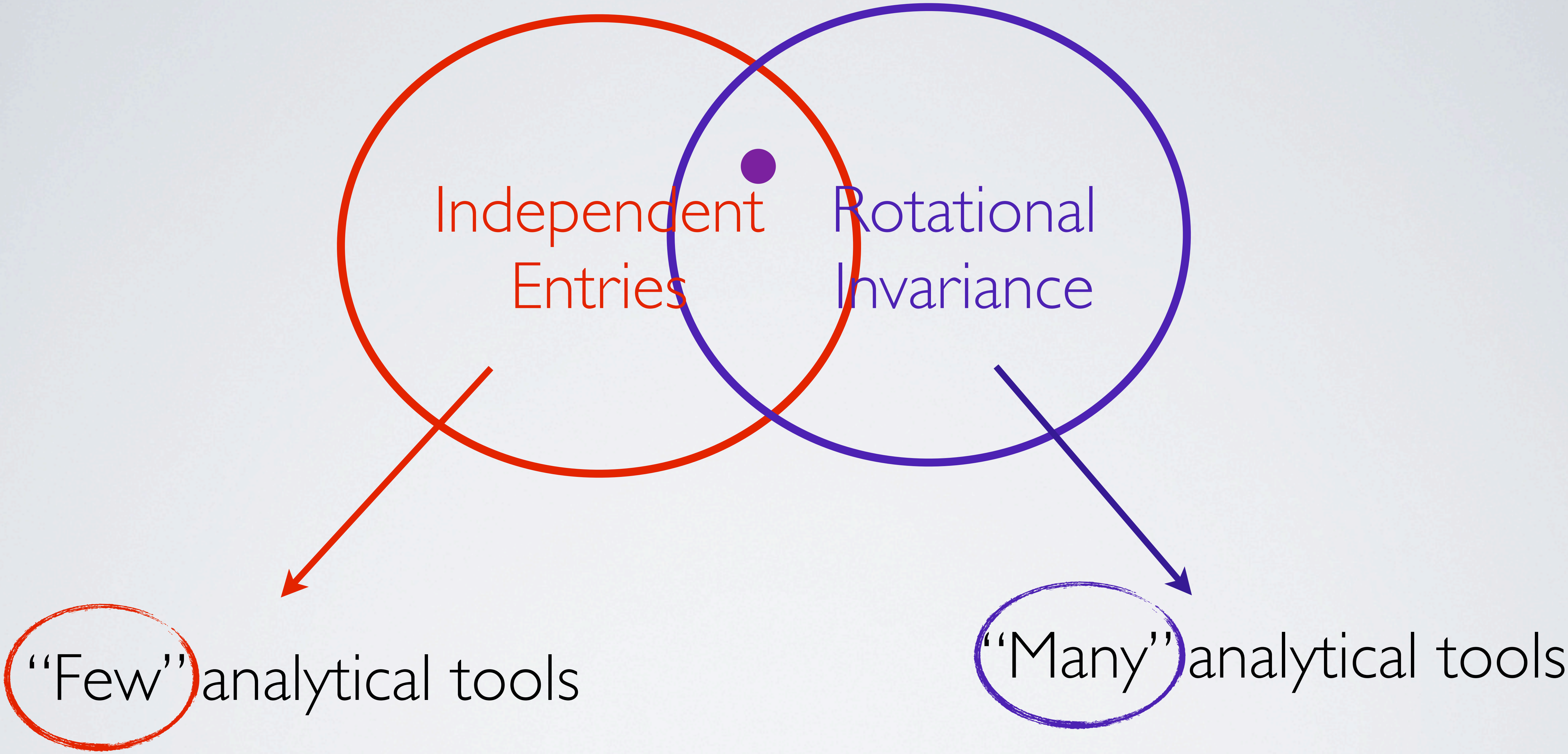


Matrices with **real** eigenvalues: a layman's classification



Given the choice between the two sets, which one would you prefer to work on?

Matrices with **real** eigenvalues: a layman's classification



Why??

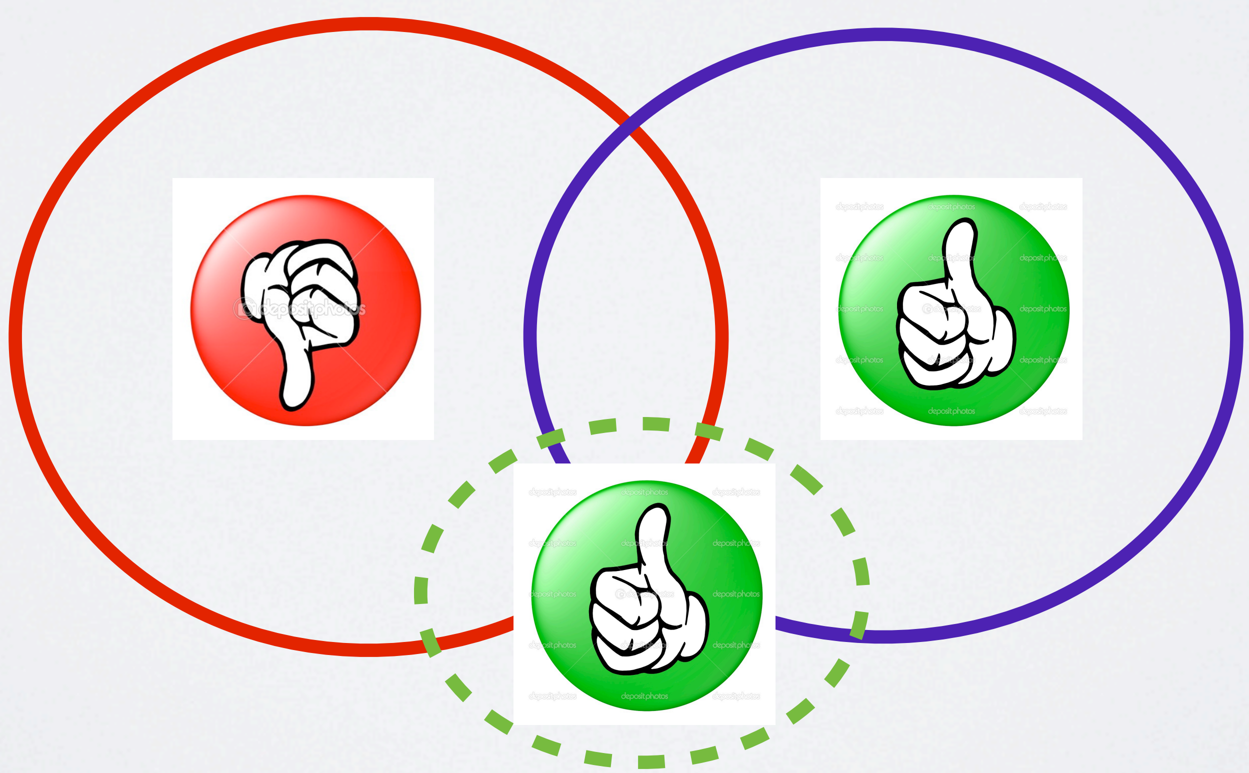
Ideally....

$$\mathcal{P}(H_{11}, \ldots, H_{NN})$$

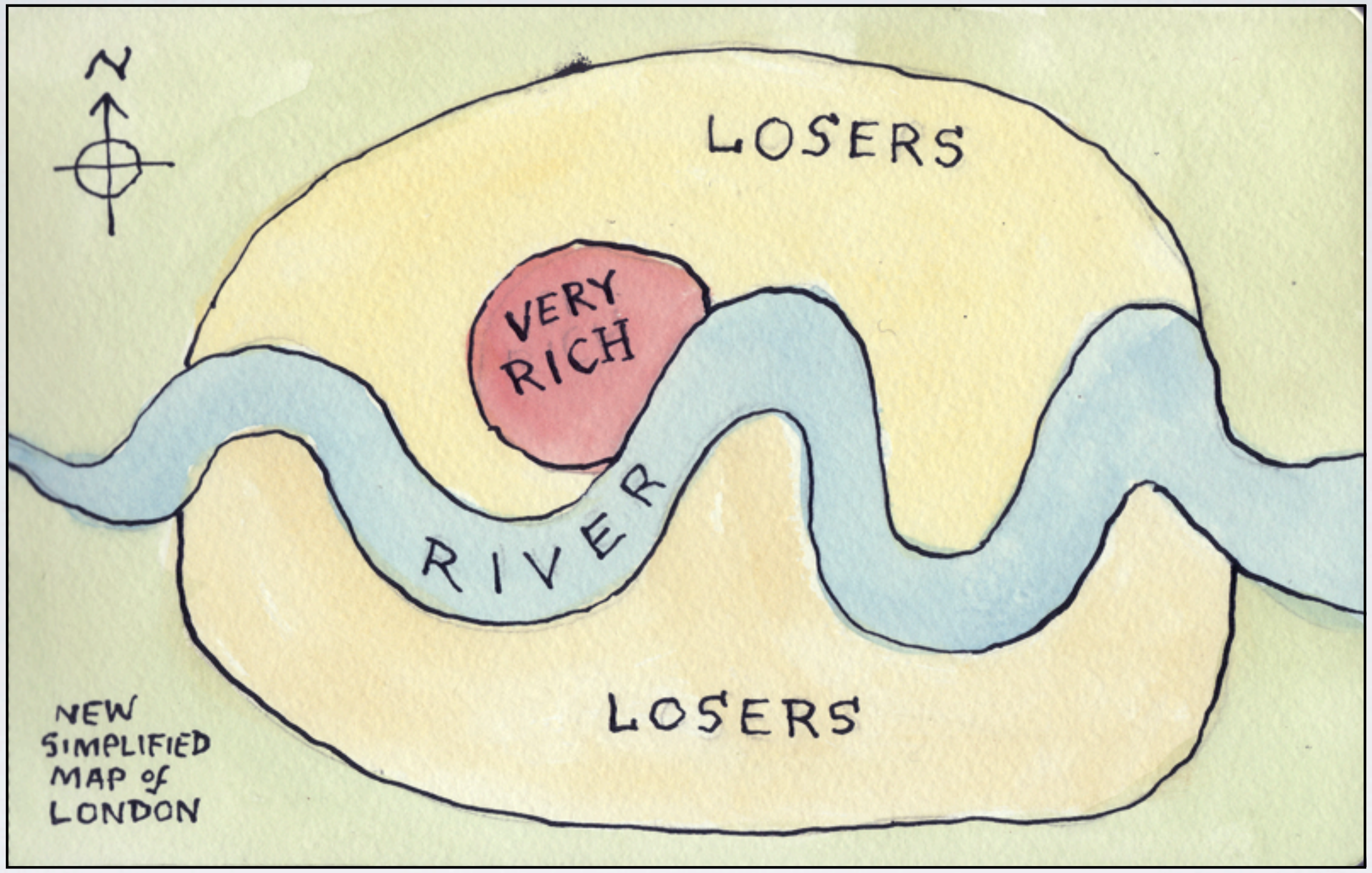
$$\downarrow$$

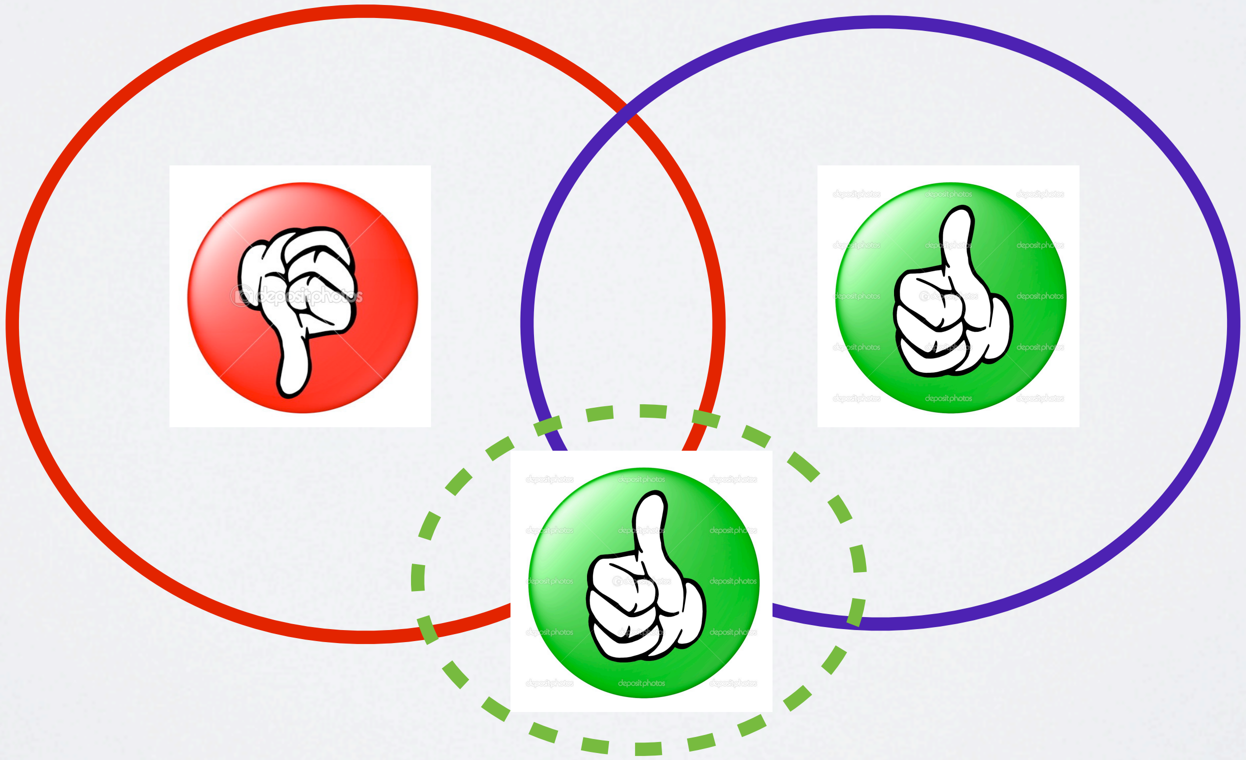
$$\mathcal{P}(\lambda_1, \ldots, \lambda_N)$$

Not always possible!



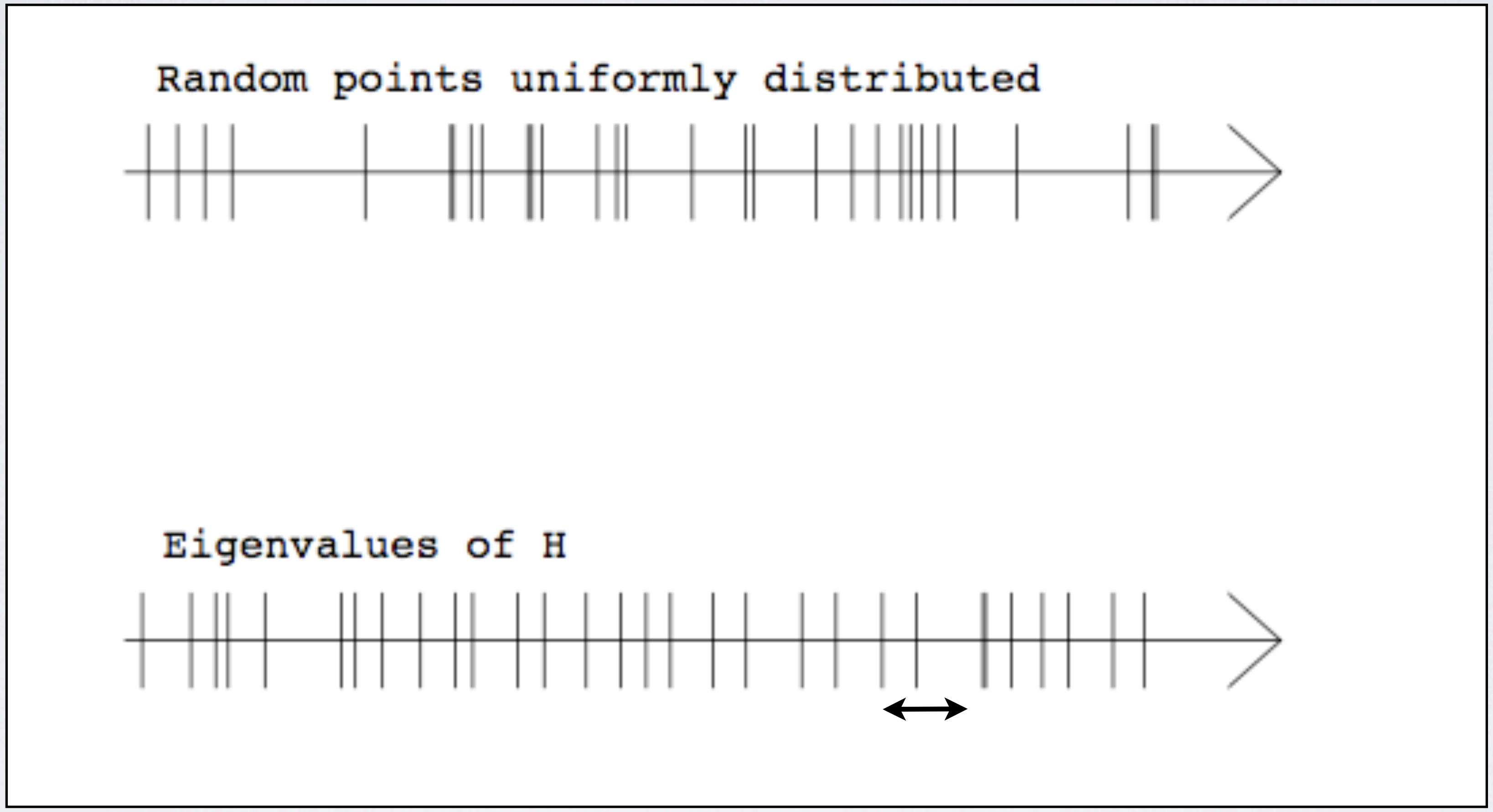
Simplified summary





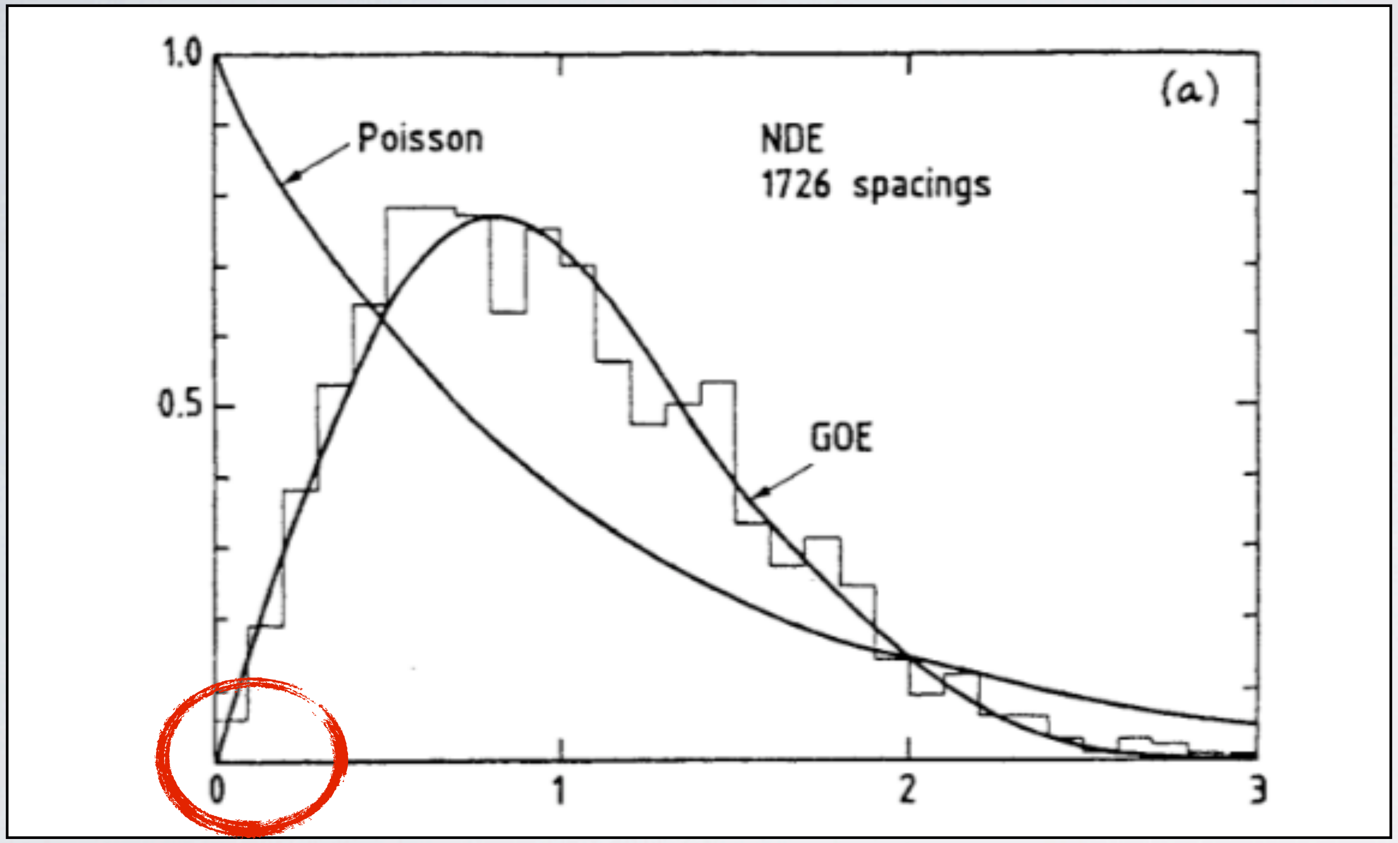
<u>Generalities</u>

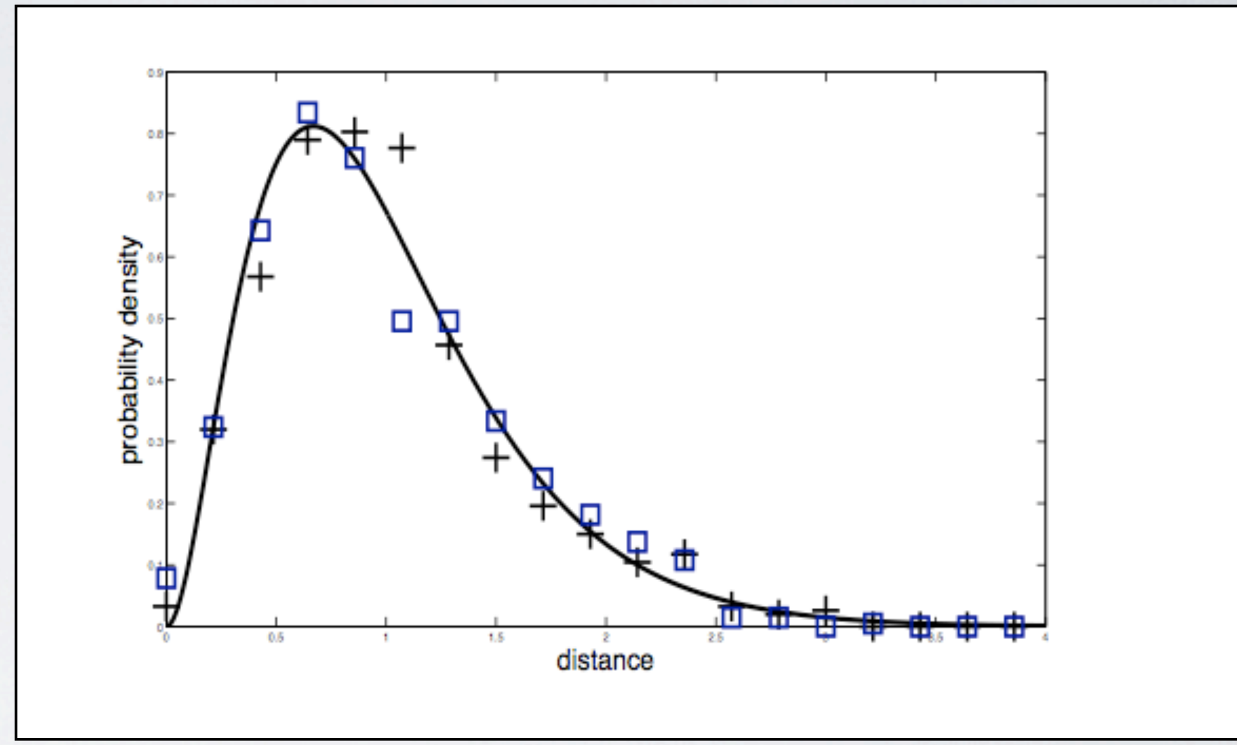
Random eigenvalues are **not** like random points on a segment



Level Repulsion

Level Spacings: universality





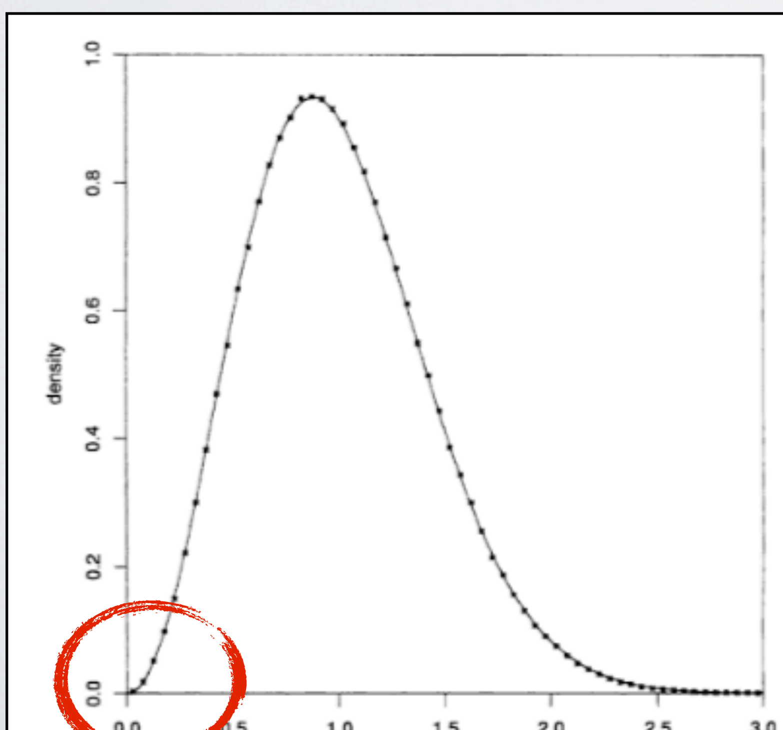


Figure 6. *A comparison of the statistics of spacings between conecutive eigenvalues of random matrices (solid curve) and nearest-neighbor spacings of zeroes of the Riemann zeta function (data points). The fit is not bad for the first million zeroes (left), but nearly perfect for the spacings among 1,041,600 zeroes near the 2×10^{20}th zero. (Figure*

$$\mathcal{P}(s) \propto s^{\beta} e^{-s^2}$$

Wigner-Dyson law

Vol. **112** (2007) *ACTA PHYSICA POLONICA A* No. 4

Proceedings of the 3rd Workshop on Quantum Chaos and Localisation Phenomena
Warsaw, Poland, May 25–27, 2007

Parking in the City

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[a]University of Hradec Králové, Hradec Kr

Physica A: Statistical Mechanics and its Applications

Volume 346, Issues 3–4, 15 February 2005, Pages 621–630

Modelling the gap size distribution of parked cars

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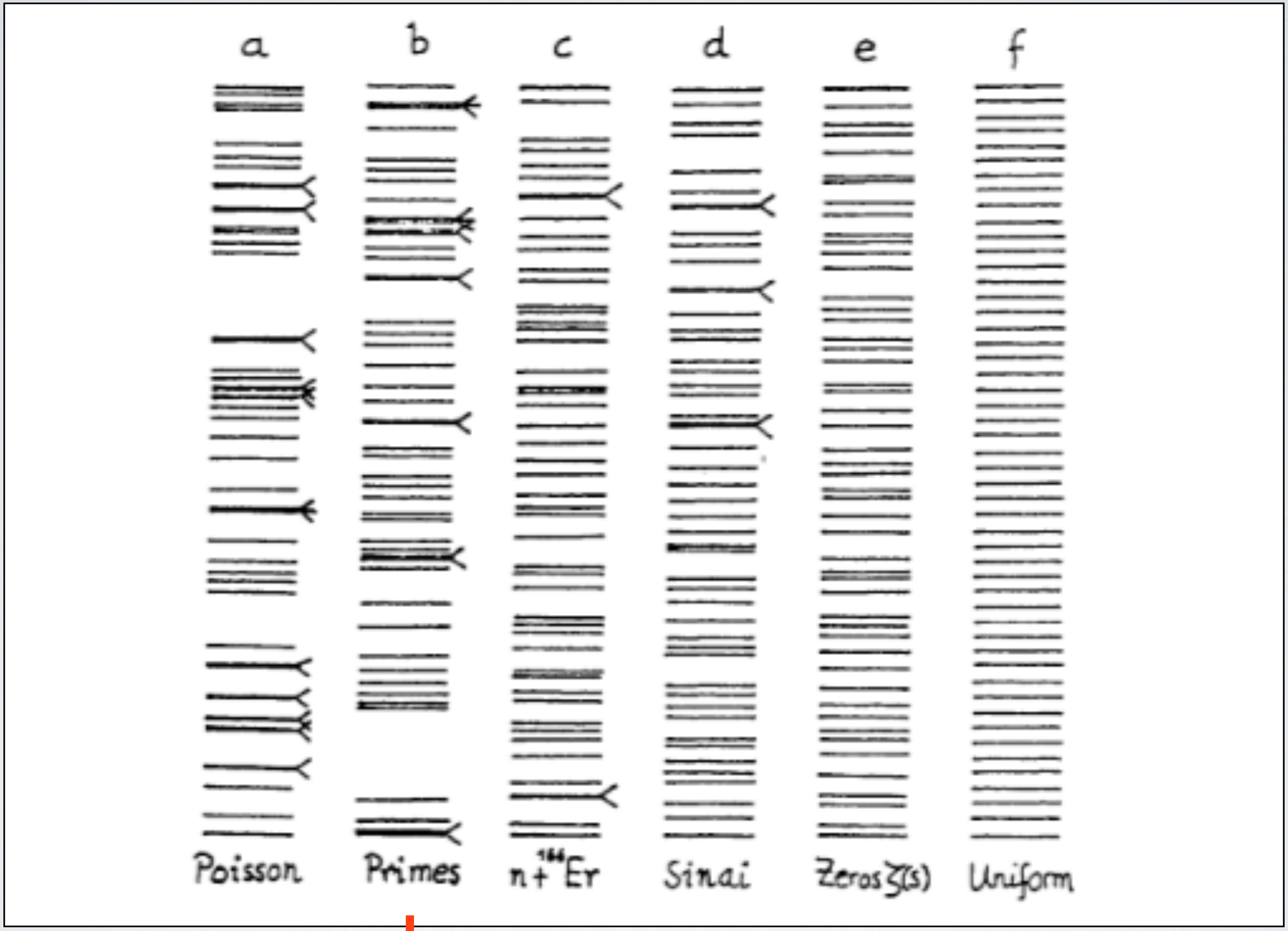
Received 19 July 2004, Available online

Modelling gap-size distribution of parked cars using random-matrix theory

A.Y. Abul-Magd
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

We apply the random-matrix theory to the car-parking problem. For this purpose, we adopt a Coulomb gas model that associates the coordinates of the gas particles with the eigenvalues of a random matrix. The nature of interaction between the particles is consistent with the tendency of the drivers to park their cars near to each other and in the same time keep a distance sufficient for manoeuvring. We show that the recently measured gap-size distribution of parked cars in a number of roads in central London is well represented by the spacing distribution of a Gaussian unitary ensemble.

PACS: 05.40; 05.20.Gg; 02.50.r; 68.43.-h
Keywords: Car parking; Coulomb gas; Gaussian unitary ensemble


a
b
c
d
e
f
Poisson
Primes
n+^{166}Er
Sinai
Zeros ζ(s)
Uniform

A digression....

Twin Prime Conjecture

DOWNLOAD *Mathematica* Notebook

There are two related conjectures, each called the twin prime conjecture. The first version states that there are an infinite number of pairs of twin primes (Guy 1994, p. 19). It is not known if there are an infinite number of such primes (Wells 1986, p. 41; Shanks 1993, p. 30), but it seems almost certain to be true. While Hardy and Wright (1979, p. 5) note that "the evidence, when examined in detail, appears to justify the conjecture," and Shanks (1993, p. 219) states even more strongly, "the evidence is overwhelming," Hardy and Wright also note that the proof or disproof of conjectures of this type "is at present beyond the resources of mathematics."

Unknown mathematician makes historical breakthrough in prime theory

Cory Doctorow at 5:40 am Tue, May 21, 2013

Yitang Zhang is a largely unknown mathematician who has struggled to find an academic job after he got his PhD, working at a Subway sandwich shop before getting a gig as a lecturer at the University of New Hampshire. He's just had a paper accepted for publication in *Annals of Mathematics*, which appears to make a breakthrough towards proving one of mathematics' oldest, most difficult, and most significant conjectures, concerning "twin" prime numbers. According to the Simons Science News article, Zhang is shy, but is a very good, clear writer and lecturer.

Now Zhang has broken through this barrier. His paper shows that there is some number N smaller than 70 million such that there are infinitely many pairs of primes that differ by N. No matter how far you go into the deserts of the truly gargantuan prime numbers — no matter how sparse the primes become — you will keep finding prime pairs that differ by less than 70 million.

Universal vs. Non-Universal

Local	Global
• Spacings • Individual Eigenvalues •	• Spectral Density • ......

However, the semicircle is quite robust....

Random sign symmetric matrix

The matrices to be considered are $2N + 1$ dimensional real symmetric matrices; N is a very large number. The diagonal elements of these matrices are zero, the non diagonal elements $v_{ik} = v_{ki} = \pm v$ have all the same absolute value but random signs. There are $\mathfrak{N} = 2^{N(2N+1)}$ such matrices. We shall calculate, after

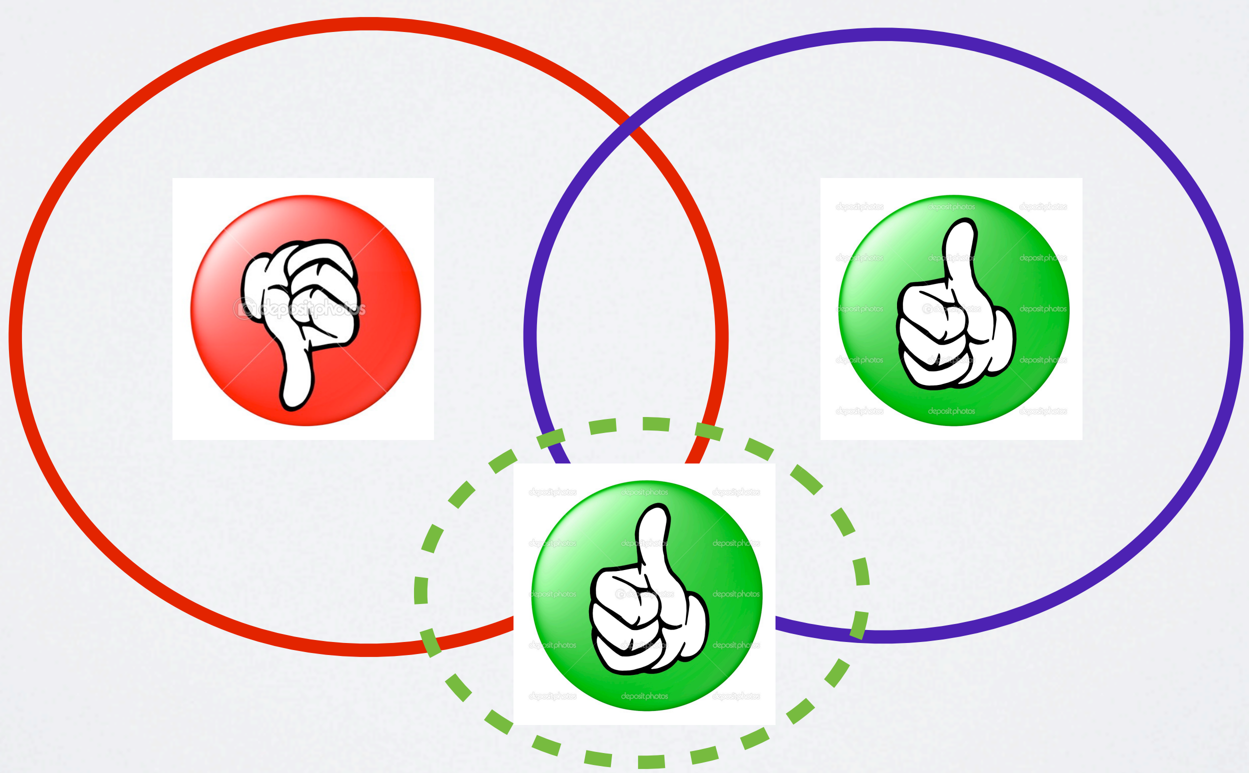
Ideally....

$$\mathcal{P}(H_{11}, \ldots, H_{NN})$$

$$\downarrow$$

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N)$$

Not always possible!



Ideally....

$$\mathcal{P}(H_{11},\ldots,H_{NN})$$

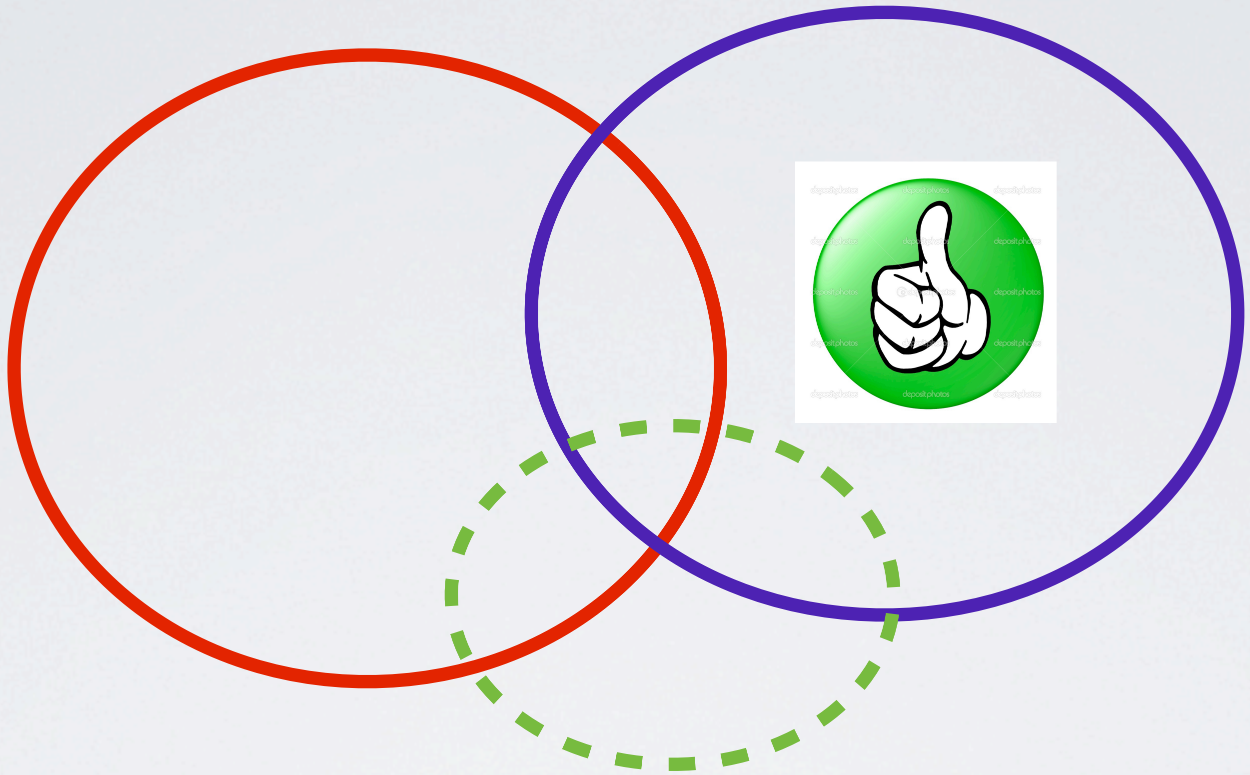
↓

$$\mathcal{P}(\lambda_1,\ldots,\lambda_N)$$

Not always possible!

Recipe?

$$\mathcal{P}(H_{11},\ldots,H_{NN}) \longrightarrow \mathcal{P}(\lambda_1,\ldots,\lambda_N)$$



$$\mathcal{P}[\mathbf{H}] = \phi\left(\mathrm{Tr}\mathbf{H},\ldots,\mathrm{Tr}\mathbf{H}^N\right)$$

Weyl's lemma

$$\mathcal{P}(\lambda_1,\ldots,\lambda_N) = C_{N,\beta}\,\phi\left(\sum_{i=1}^{N}\lambda_i,\ldots,\sum_{i=1}^{N}\lambda_i^N\right)\prod_{j<k}|\lambda_j-\lambda_k|^\beta$$

$$\mathcal{P}(H_{11},\ldots,H_{NN}) \propto \prod_i e^{-H_{ii}^2/2\sigma^2} \prod_{j>k} e^{-2H_{jk}^2/2\sigma^2}$$
$$\propto e^{-\frac{1}{2\sigma^2}\mathrm{Tr}(\mathbf{H}^2)}$$

ON THE DISTRIBUTION OF ROOTS OF CERTAIN DETERMINANTAL EQUATIONS

By P. L. HSU [1939]

256 ROOTS OF DETERMINANTAL EQUATIONS

Theorem 2. *If the $\frac{1}{2}p(p+1)$ variables $s_{ij}(i \leqslant j = 1, 2, \ldots, p)$ have such a domain of existence that the symmetric matrix $\|s_{ij}\|$ is always non-singular, and if they are so distributed that their joint probability density function depends only on the latent roots, say $\lambda_1, \lambda_2, \ldots, \lambda_p$, arranged in the order of descending magnitude, of $\|s_{ij}\|$, i.e. if*

$$df = g(\lambda_1, \lambda_2, \ldots, \lambda_p) \prod ds_{ij},$$

then the joint distribution law of the λ_i is the following:

$$\pi^{\frac{1}{4}p(p+1)}\left\{\prod_{i=1}^{p} \Gamma\tfrac{1}{2}(p-i+1)\right\}^{-1}\left\{\prod_{i=1}^{p}\prod_{j=i+1}^{p}(\lambda_i-\lambda_j)\right\} g(\lambda_1, \ldots, \lambda_p) \prod d\lambda. \quad \ldots\ldots(24)$$

Level Repulsion!

Proof. It is a familiar argument that the general formula (24) will follow if we can find

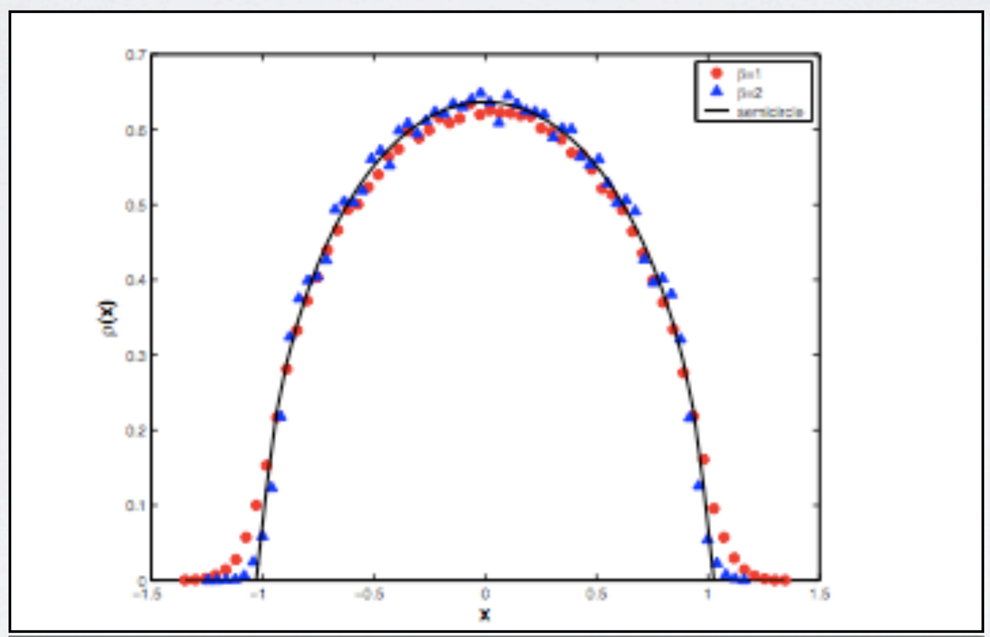
Rotationally Invariant Models

$$\mathcal{P}_\beta(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_{N,\beta}} e^{-\frac{1}{2}\sum_{i=1}^N \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

$\beta = 1, 2, 4$

Confinement
(non-universal)

Level Repulsion
(universal)



Strongly Correlated Random Variables!!

Vandermonde determinant

$$\prod_{i<j}^{N}(\lambda_i - \lambda_j) = (-1)^{\frac{N(N-1)}{2}} \det \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_N \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \lambda_1^{N-1} & \dots & \lambda_N^{N-1} \end{pmatrix}$$

Very funny properties...

It is instructive to consider the 2×2 case. There we have for instance:

$$(\lambda_1 - \lambda_2) = \frac{-1}{3 \cdot 5} \det \begin{pmatrix} 3 & 3 \\ 2 + 5\lambda_1 & 2 + 5\lambda_2 \end{pmatrix} = \frac{-1}{5 \cdot \sqrt{2}} \det \begin{pmatrix} 5 & 5 \\ 3 + \sqrt{2}\lambda_1 & 3 + \sqrt{2}\lambda_2 \end{pmatrix}$$

$$\prod_{i<j}^{N}(\lambda_i - \lambda_j) = \frac{(-1)^{\frac{N(N-1)}{2}}}{a_0 a_1 \dots a_{N-1}} \det \begin{pmatrix} \pi_0(\lambda_1) & \dots & \pi_0(\lambda_N) \\ \pi_1(\lambda_1) & \dots & \pi_1(\lambda_N) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \pi_{N-1}(\lambda_1) & \dots & \pi_{N-1}(\lambda_N) \end{pmatrix}$$

Arbitrary polynomials...

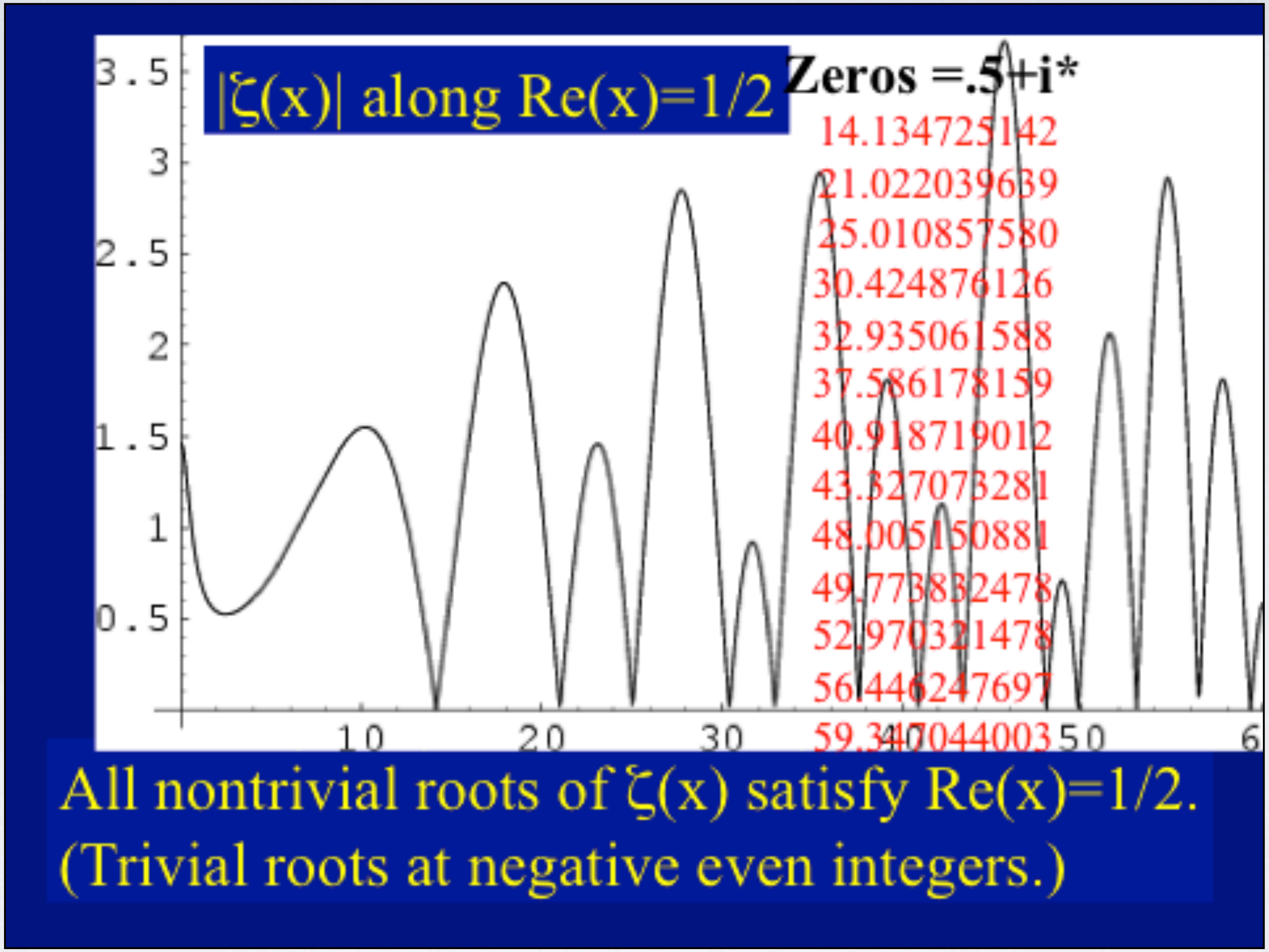
$$\pi_k(\lambda_i) = a_k \lambda_i^k + $$

A few modern applications of RMT

The Riemann Hypothesis

$$\zeta(x) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{u^{x-1}}{e^u - 1}du = \sum_{k=1}^{\infty}\frac{1}{k^x}$$

All nontrivial roots of ζ(x) satisfy Re(x)=1/2.
(Trivial roots at negative even integers.)


|ζ(x)| along Re(x)=1/2
Zeros =.5+i*
14.134725142
21.022039639
25.010857580
30.424876126
32.935061588
37.586178159
40.918719012
43.327073281
48.005150881
49.773832478
52.970321478
56.446247697
59.347044003
3.5
3
2.5
2
1.5
1
0.5
10
20
30
40
50
6
All nontrivial roots of ζ(x) satisfy Re(x)=1/2.
(Trivial roots at negative even integers.)

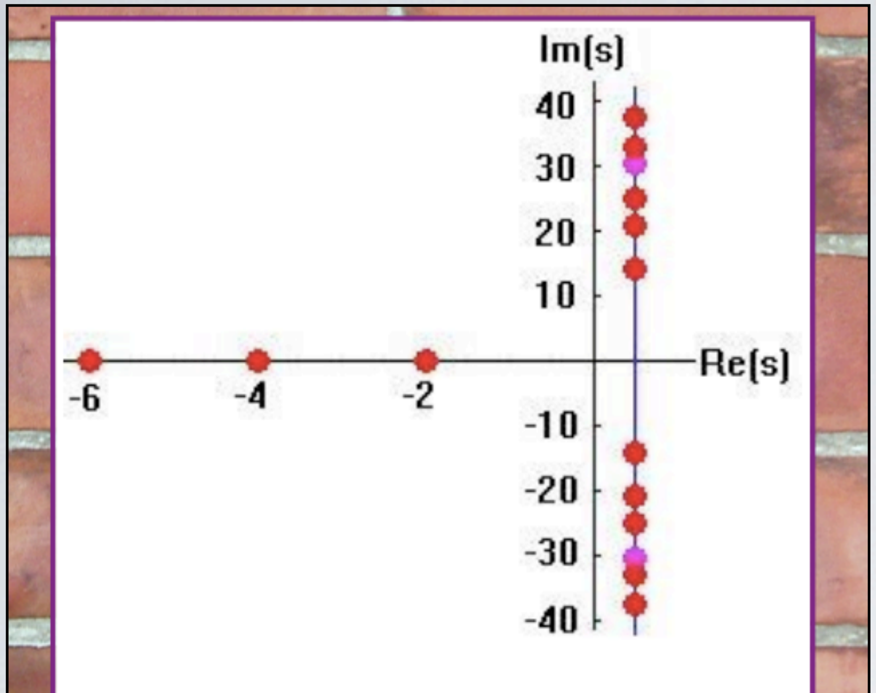


Figure 1: THE ZEROS OF THE ζ-FUNCTION.

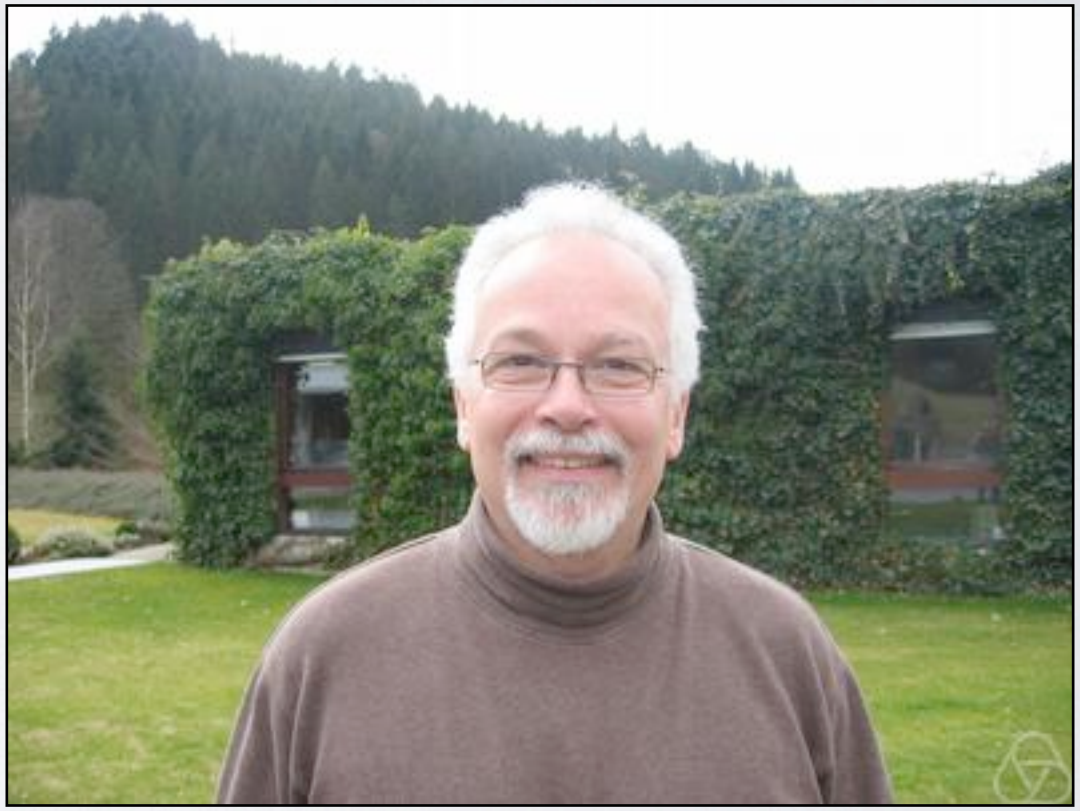
$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

liegenden Wurzeln von $\xi(t) = 0$ multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.

...it is very probable that all roots are real. One would, however, wish for a strict proof of this; I have, though, after some fleeting futile attempts, provisionally put aside the search for such, as it appears unnecessary for the next objective of my investigation.

"Sometimes I think that we essentially have a complete proof of the Riemann Hypothesis except for a gap. The problem is, the gap occurs right at the beginning, and so it's hard to fill that gap because you don't see what's on the other side of it."

Hugh Lowell Montgomery



Montgomery's Pair Correlation Conjecture

Montgomery's pair correlation conjecture, published in 1973, asserts that the two-point correlation function $R_2(r)$ for the zeros of the Riemann zeta function $\zeta(z)$ on the critical line is

$$R_2(r) = 1 - \frac{\sin^2(\pi r)}{(\pi r)^2}.$$

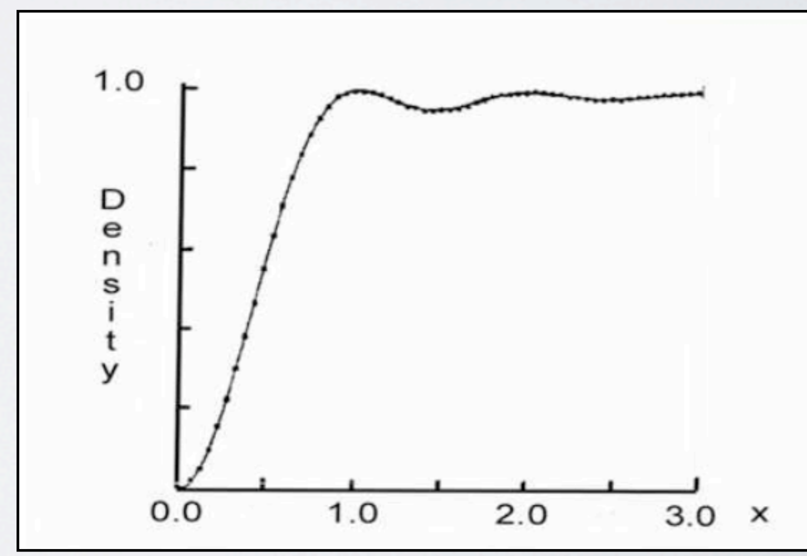
As first noted by Dyson, this is precisely the form expected for the pair correlation of random Hermitian matrices (Derbyshire 2004, pp. 287-291).

In 1972, Hugh Montgomery, a number theorist at the University of Michigan, was visiting the Institute for Advanced Study. Montgomery had been studying the distribution of zeroes of the zeta function, in hopes of gaining insight into the Riemann Hypothesis. He was able to prove that the Riemann Hypothesis had implications for the spacing of zeroes along the critical line, but his key discovery was an additional property that the zeroes seemed to have, one which implied a particularly nice formula for the average spacing between zeroes.

Odlyzko's computations agree amazingly well with Montgomery's conjecture.

During tea one day at the Institute, Montgomery was introduced to Dyson and described his conjecture. Dyson immediately recognized it as the same result as had been obtained for random matrices.

"It just so happened that he was one of the two or three physicists in the world who had worked all of these things out, so I was actually talking to the greatest expert in exactly this!" Montgomery recalls.



Nearest Neighbor Spacings & Pairwise Correlation Functions

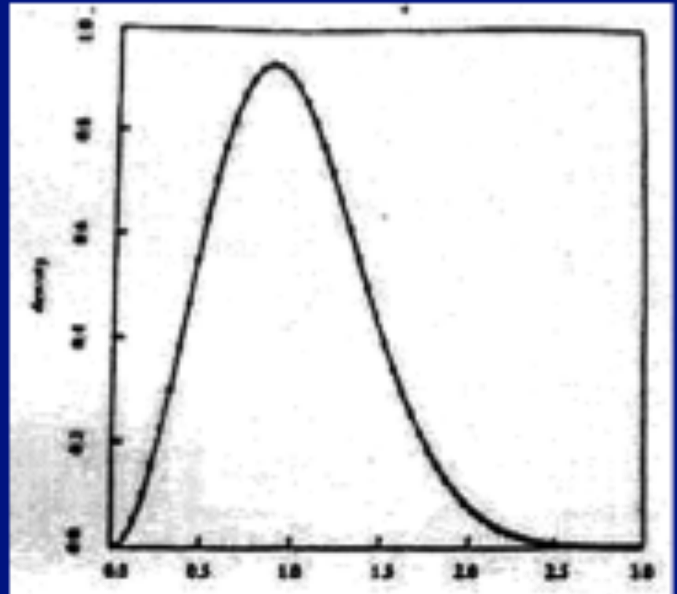


FIGURE 1
Nearest neighbor spacings among 70 million zeroes beyond the 10^{20}-th zero of zeta, versus $\mu_1(GUE)$

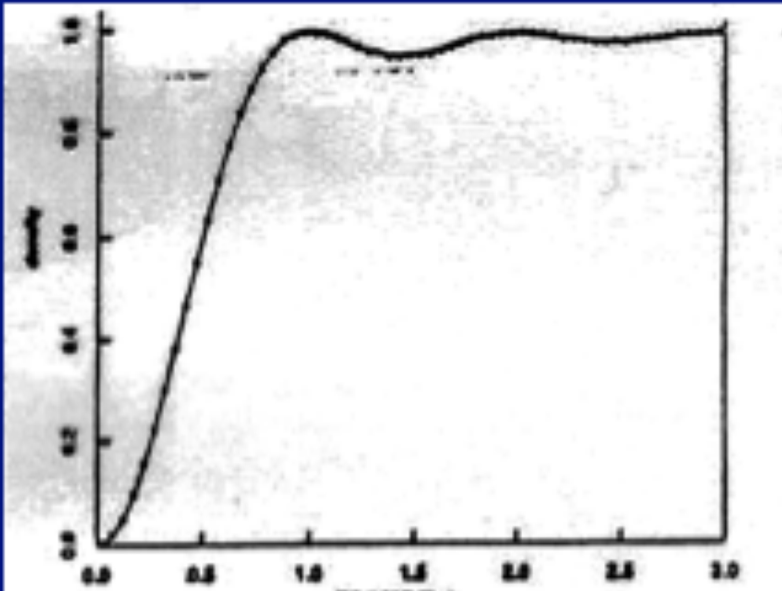


FIGURE 2
Pair correlation for zeros of zeta based on 8×10^{6} zeros near the 10^{20} zero, versus the GUE conjectured density $1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$.

Simple Proof of Riemann's Hypothesis of the Zeta function

by [redacted], 6/20/2006

The Zeta function is defined as:

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\ldots\ s\neq 1\ (k=1\to\infty)$$

Hardy, 1999 showed that:

$$0=\xi(1-s)=\xi(s)\ \text{where}\ \xi(s)\ \text{extends}\ \zeta(s)\ \text{into}$$

$$\text{for all}\ 0<s<1\ \text{at the "non-trivial" zeros}$$

$$0=\sum_{k=1}^{\infty}\frac{1}{k^{(1-s)}}=\sum_{k=1}^{\infty}\frac{1}{k^s}\ \text{for all}\ 0<s<1,\ \text{positiv}$$

$$0=\sum_{k=1}^{\infty}\frac{1}{k^s}-\sum_{k=1}^{\infty}\frac{1}{k^{(1-s)}}$$

$$0=\sum_{k=1}^{\infty}\left(\frac{1}{k^s}-\frac{1}{k^{(1-s)}}\right)$$

$$0=\sum_{k=1}^{\infty}\left(\frac{k^{(1-s)}-k^s}{k^s k^{(1-s)}}\right)$$

$$0=\sum_{k=1}^{\infty}\left(\frac{k^{(1-s)}-k^s}{k}\right)\Rightarrow 0=\sum_{k=1}^{\infty}\left(\frac{0}{k}\right)\quad if\ \exists\ s$$

$$\Rightarrow k^{(1-s)}=k^s\Rightarrow(1-s)=s\quad for\ k=1\to\infty,\ 0<s<1$$

$$\text{substituting complex}\ s\ (1-(\sigma+ix))=(\sigma+ix)$$

$$\text{for real}\ \sigma=1-\sigma\Rightarrow 2\sigma=1$$

$$\Rightarrow\sigma=\frac{1}{2}\Rightarrow\sigma+ix=\frac{1}{2}-ix\Rightarrow\Re[s]=\frac{1}{2}\ \ for\ k=1\to\infty,\ 0<s<1,\ \text{'critical strip'}$$

QED

Hardy, G. H. Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed. New York: Chelsea, 1999.

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[0807.0090] A proof of the Riemann hypothesis

http://arxiv.org/abs/0807.0090

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Mathematics > Number Theory

A proof of the Riemann hypothesis

Xian-Jin Li

(Submitted on 1 Jul 2008 (v1), last revised 6 Jul 2008 (this version, v4))

This paper has been withdrawn by the author, due to a mistake on page 29.

Comments: withdrawn by author, due to mistake on page 29
Subjects: Number Theory (math.NT)
MSC classes: 11M26
Cite as: arXiv:0807.0090v4 [math.NT]

Submission history

From: Xian-Jin Li [view email]
[v1] Tue, 1 Jul 2008 19:43:13 GMT (20kb)
[v2] Wed, 2 Jul 2008 11:05:52 GMT (20kb)
[v3] Thu, 3 Jul 2008 03:44:03 GMT (20kb)
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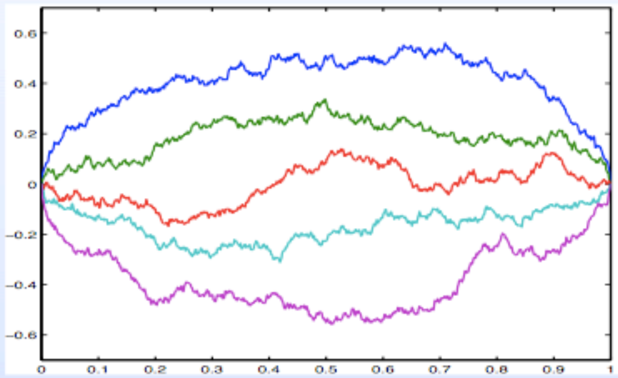
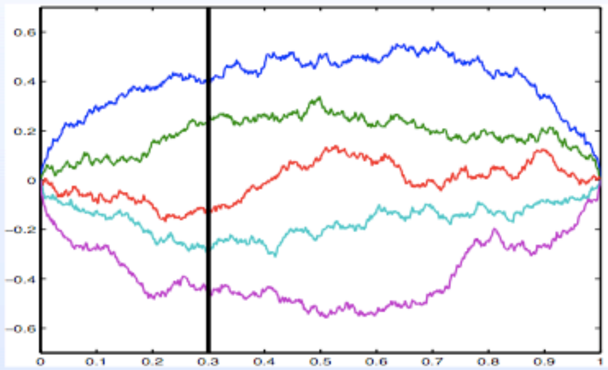
Non-intersecting Brownian motion paths

- Take n independent 1-dimensional Brownian motions with time in $[0,1]$ conditioned so that:
  - All paths start and end at the same point.
  - The paths **do not intersect** at any intermediate time.



Five non-intersecting Brownian bridges

- **Remarkable fact**: At any intermediate time the positions of the paths have **exactly the same distribution** as the eigenvalues of an $n \times n$ GUE matrix (up to a scaling factor).



Positions of five non-intersecting Brownian paths behave the same as the eigenvalues of a 5×5 GUE matrix

- This interpretation is basic for the connection of random matrix theory with growth models of statistical physics.

Introduction. Since the pioneering work of de Gennes [1], followed up by Fisher [2], the subject of vicious (non-intersecting) random walkers has attracted a lot of interest among physicists. It has been studied in the context of wetting and melting [2], networks of polymers [3] and fibrous structures [1], persistence properties in nonequilibrium systems [4] and stochastic growth models [5, 6]. There also exist connections between the

PHYSICAL REVIEW E VOLUME 52, NUMBER 6 DECEMBER 1995

Vicious walkers and directed polymer networks in general dimensions

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A. J. Guttmann
Department of Mathematics, The University of Melbourne, Parkville, Victoria 3052, Australia

Covariance Matrices

$$
X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{cc} \text{phys.} & \text{math} \\ \hline x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{33} \end{array}
\qquad \begin{array}{c} \text{in general} \\ (\text{MxN}) \end{array}
$$

$$
X^t = \begin{vmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{33} \end{vmatrix}
\qquad \begin{array}{c} \text{in general} \\ (\text{NxM}) \end{array}
$$

$$
W = X^t X = \begin{vmatrix} x_{11}^2 + x_{21}^2 + x_{31}^2 & x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{33} \\ x_{12}x_{11} + x_{22}x_{21} + x_{33}x_{31} & x_{12}^2 + x_{22}^2 + x_{33}^2 \end{vmatrix}
$$

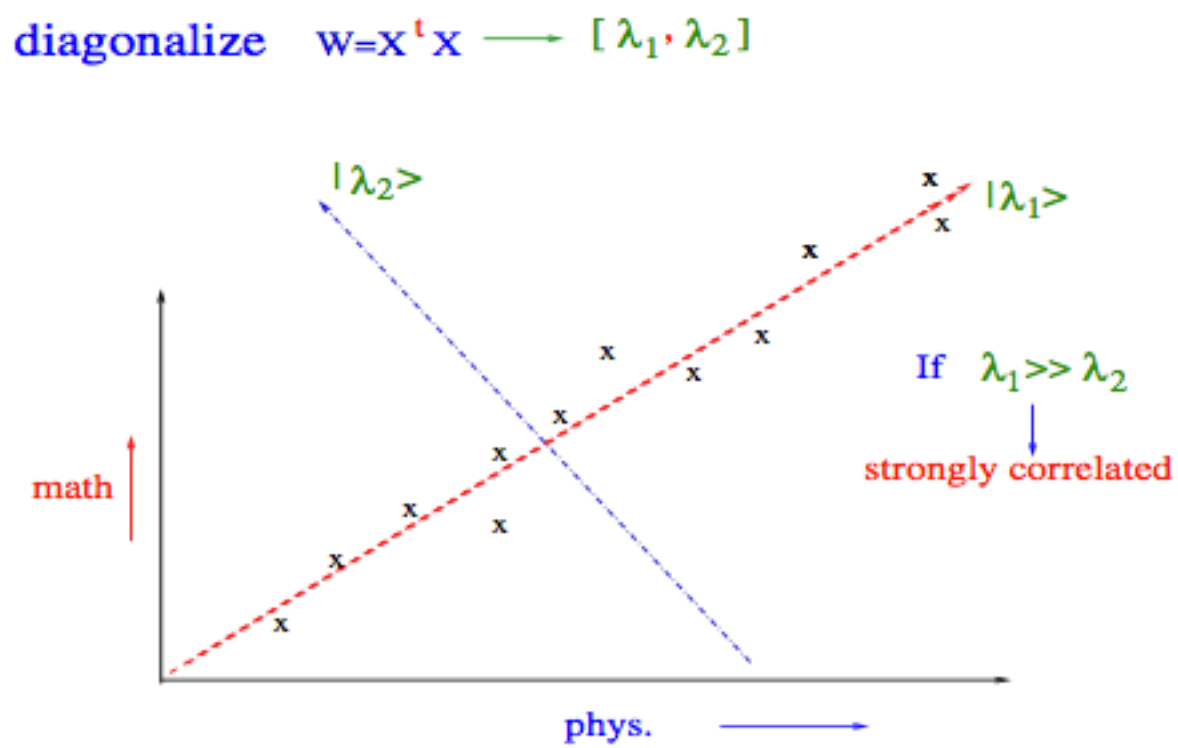
→ (NxN) COVARIANCE MATRIX (unnormalized)

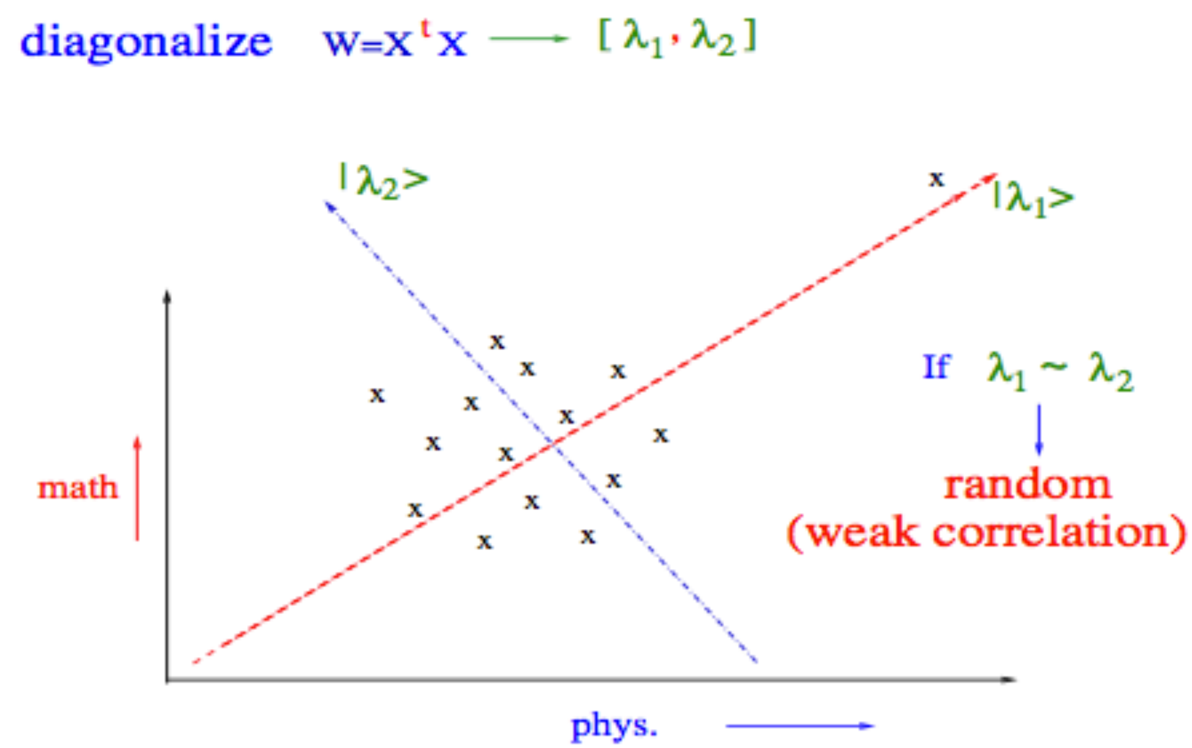
[borrowed from S.N. Majumdar,
"Top eigenvalue of a random matrix: a tale of tails."]

Principal Component Analysis

Consider N students and $M = 2$ subjects (phys. and math.)

$X \rightarrow (N \times 2)$ matrix and $W = X^t X \rightarrow 2 \times 2$ matrix



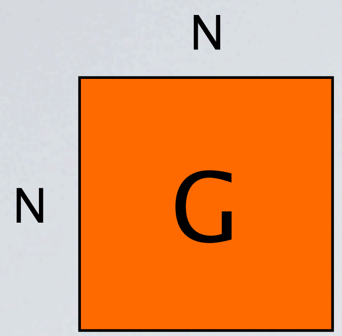


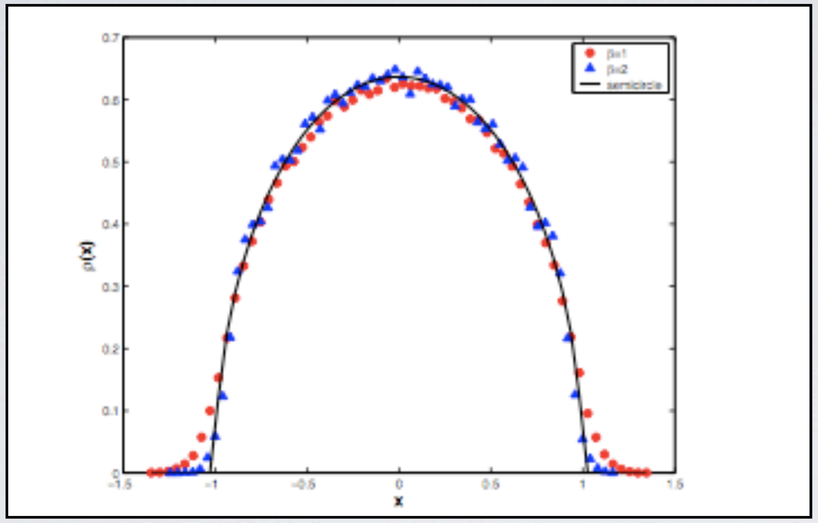
data compression via 'Principal Component Analysis' (PCA)

$\longrightarrow$ practical method for image compression in computer vision

Null model $\rightarrow$ random data: $X \rightarrow$ random $(M \times N)$ matrix

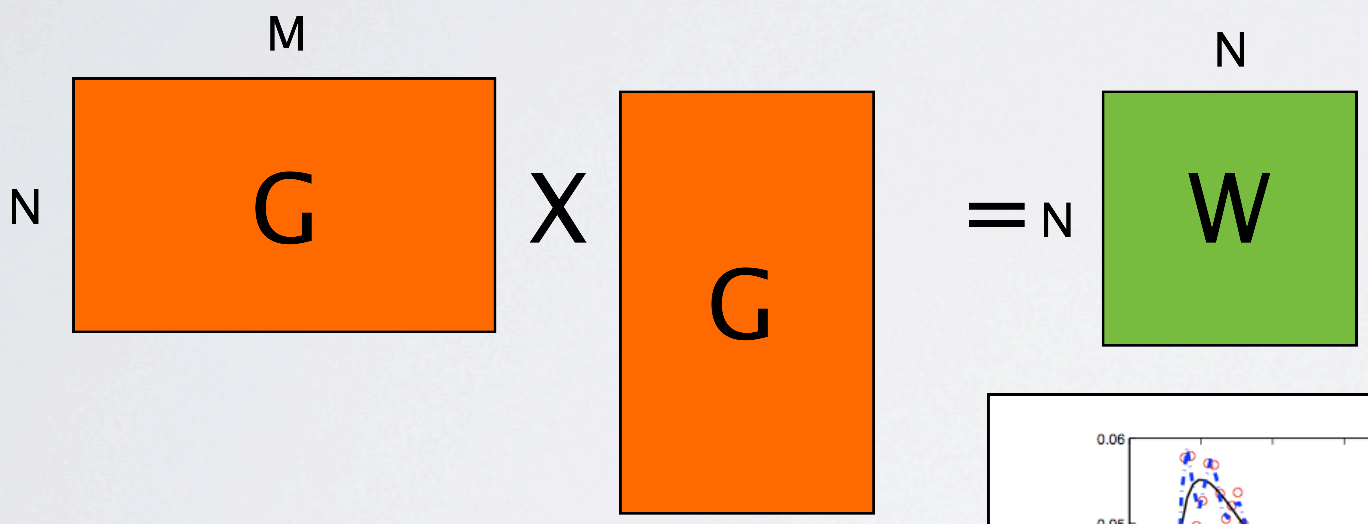
$\rightarrow W = X^t X \rightarrow$ random $N \times N$ matrix (Wishart, 1928)





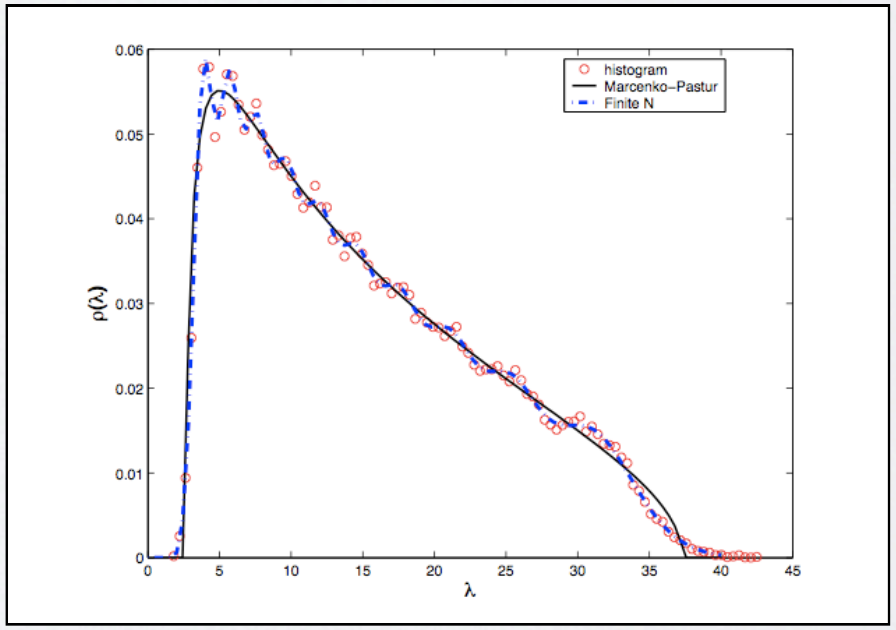
Gaussian

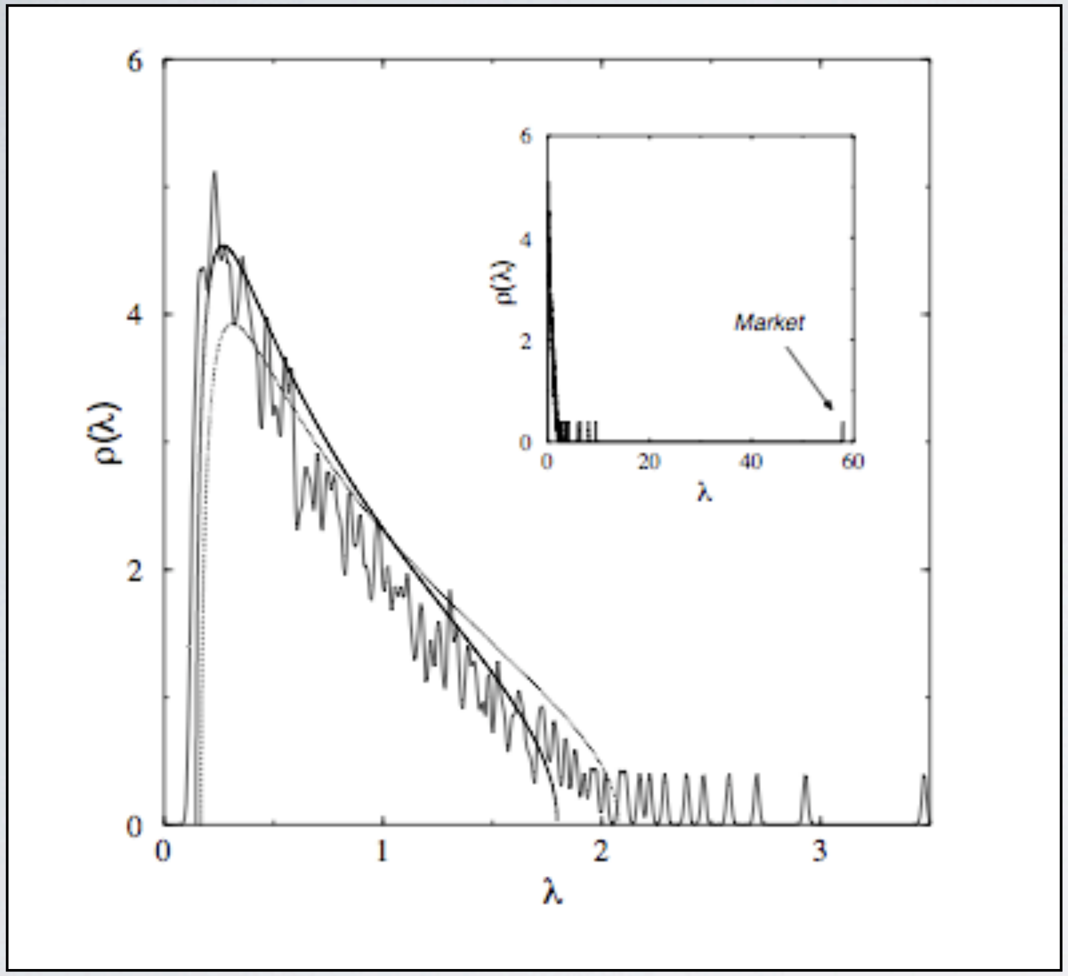
$$\mathbf{G}_{ij} \sim \mathcal{N}(0,1)$$



Wishart

$$\mathbf{W} = \mathbf{G}\mathbf{G}^{\dagger}$$





VOLUME 83, NUMBER 7 PHYSICAL REVIEW LETTERS 16 AUGUST 1999

Noise Dressing of Financial Correlation Matrices

Laurent Laloux,[1,*] Pierre Cizeau,[1] Jean-Philippe Bouchaud,[1,2] and Marc Potters[1]
[1]*Science & Finance, 109-111 rue Victor Hugo, 92532 Levallois Cedex, France*
[2]*Service de Physique de l'État Condensé, Centre d'études de Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France*

time. From this point of view, it is interesting to compare the properties of an empirical correlation matrix **C** to a null hypothesis purely *random* matrix as one could obtain from a finite time series of strictly independent assets. Deviations from the random matrix case might then suggest the presence of true information. The theory of random matri-

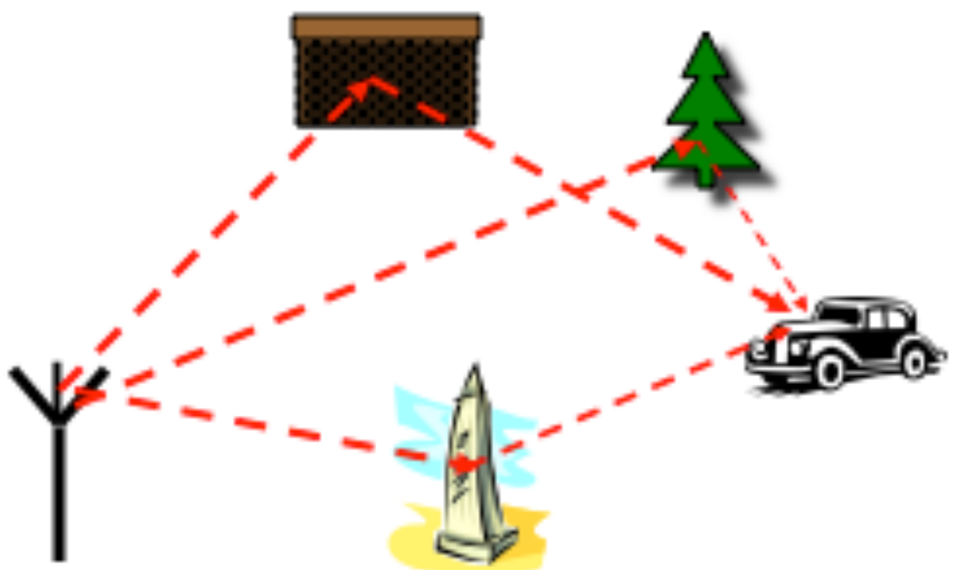
Debate: is the bulk of the stock market correlation matrix just pure noise?

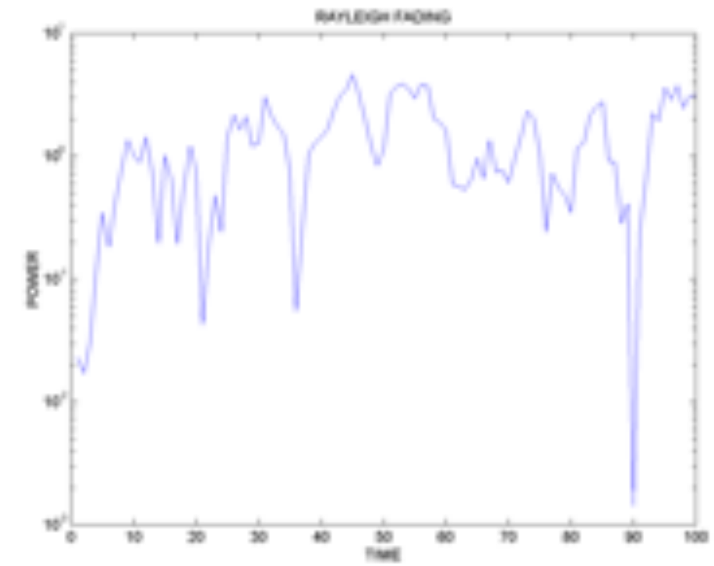
INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **36** (2003) 3009–3032 PII: S0305-4470(03)37794-7

A new method to estimate the noise in financial correlation matrices

Thomas Guhr[1] and Bernd Kälber[2,3]

Multipath Wireless Channels





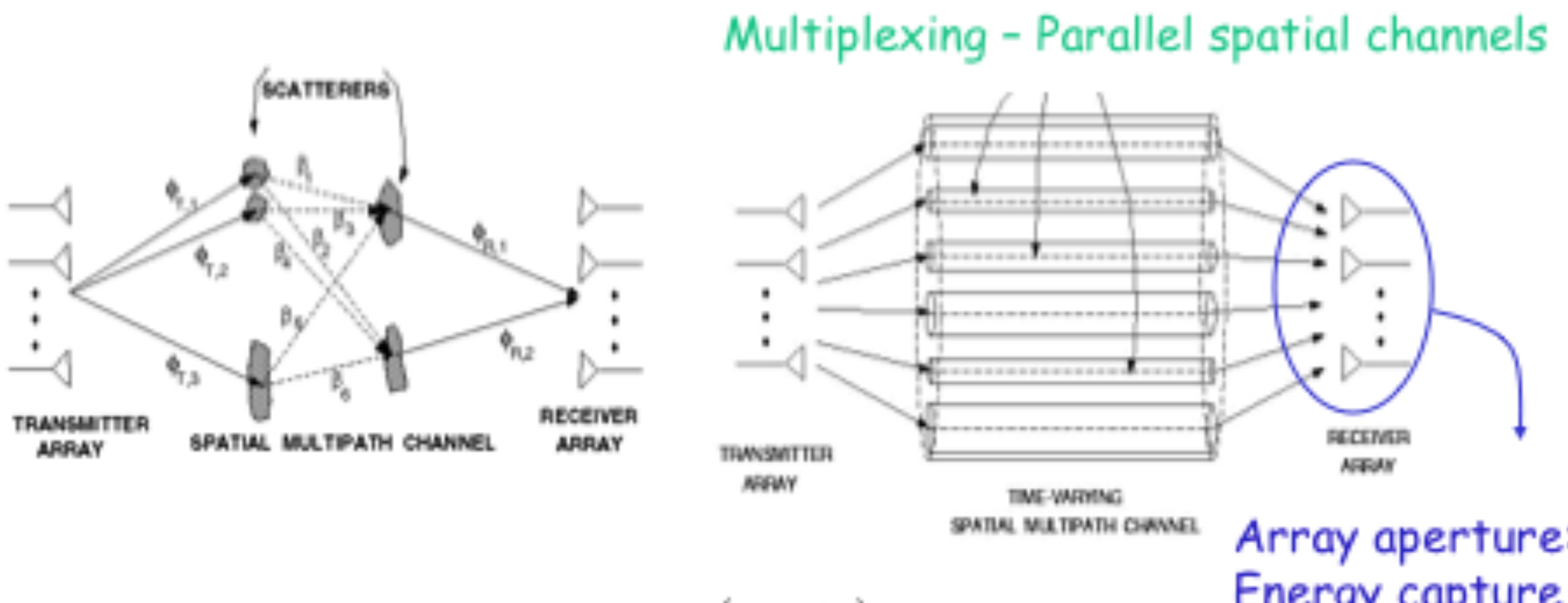
- Multipath signal propagation over spatially distributed paths due to signal scattering from multiple objects
  - Necessitates statistical channel modeling
  - Accurate and analytically tractable → Understanding the physics!

- Fading – fluctuations in received signal strength
- Diversity – statistically independent modes of communication

Capacity of Multi-antenna Gaussian Channels

İ. Emre Telatar*

Antenna Arrays: Multiplexing and Energy Capture



which the received vector $\boldsymbol{y} \in \mathbb{C}^r$ depends on the transmitted vector $\boldsymbol{x} \in \mathbb{C}^t$ via

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{n} \tag{1}$$

where H is a $r \times t$ complex matrix and $\boldsymbol{n}$ is zero-mean complex Gaussian noise with

We will consider several scenarios for the matrix H:

1. H is deterministic.
2. H is a random matrix (for which we shall use the notation $\boldsymbol{H}$), chosen according to a probability distribution, and each use of the channel corresponds to an independent realization of $\boldsymbol{H}$.
3. H is a random matrix, but is fixed once it is chosen.

related to the "mutual information" between the senders and the receivers

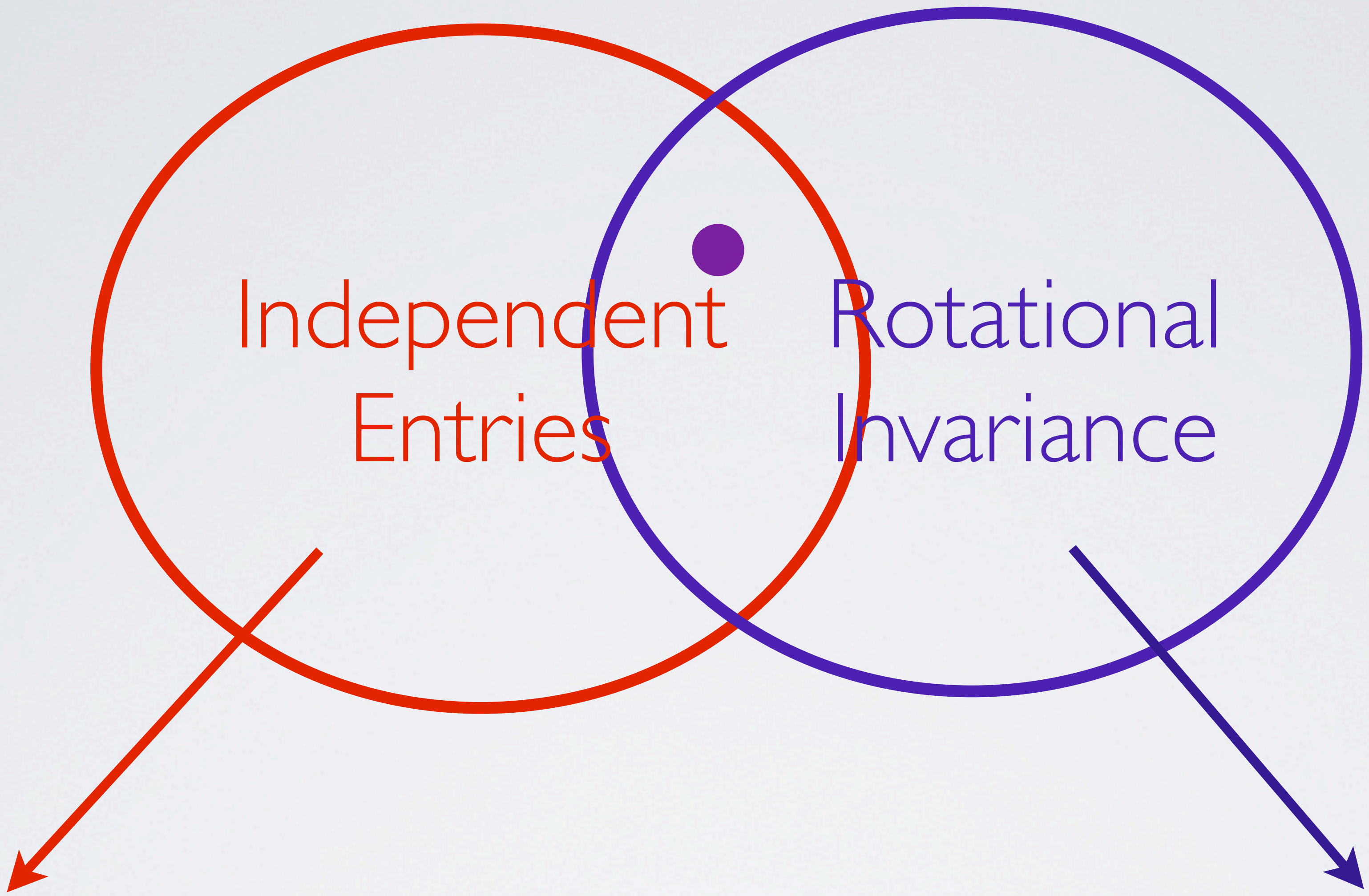
4.2 Evaluation of the Capacity

Although the expectation $\mathcal{E}\left[\log\det\left(I_r + (P/t)\boldsymbol{H}\boldsymbol{H}^\dagger\right)\right]$ is easy to evaluate for either

Wishart matrices!!

Techniques

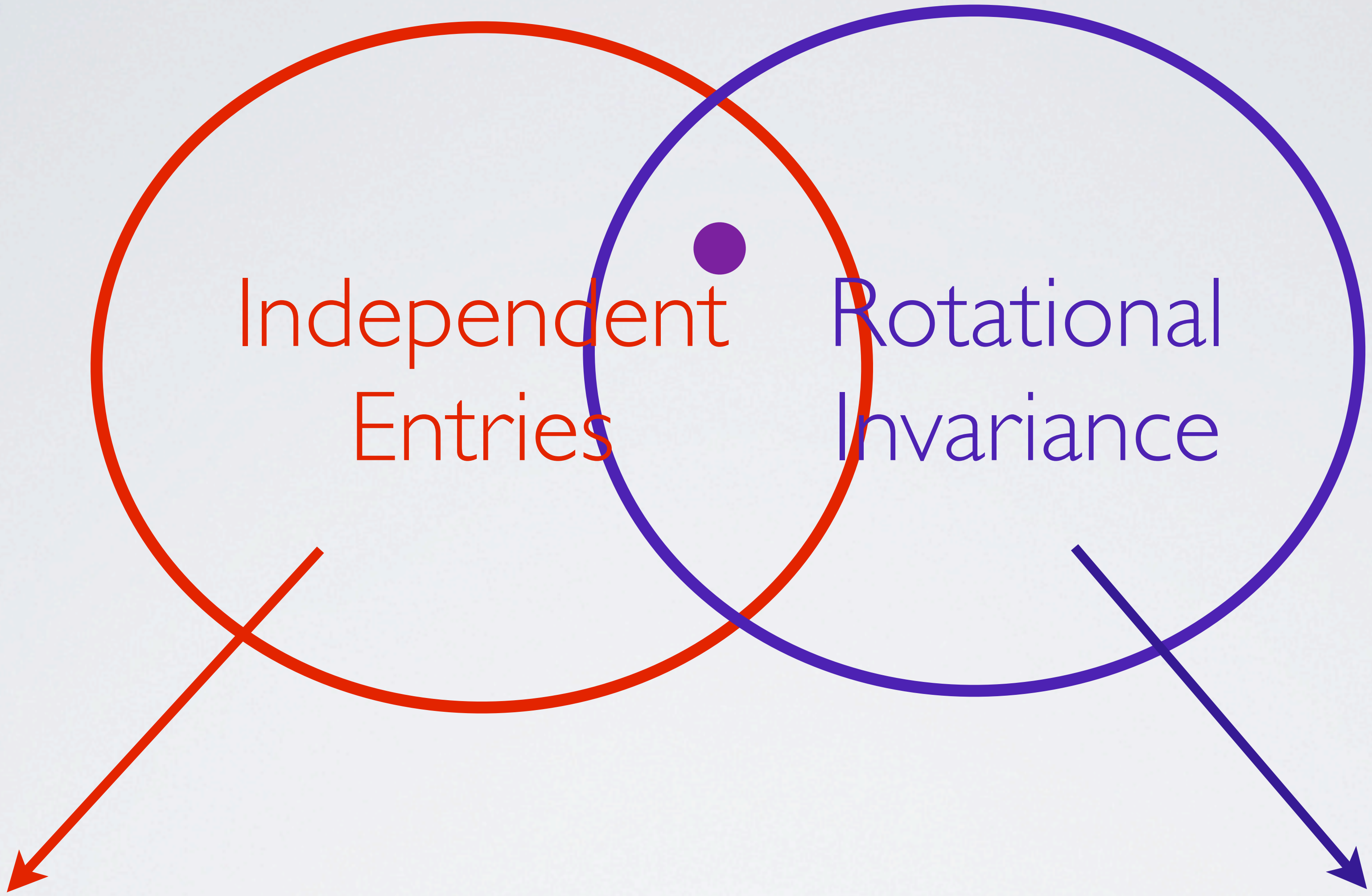
Techniques



- Edwards-Jones formula
- Moments method

- Andreief formula
- Orthogonal Polynomials
- Coulomb gas

Techniques



- Edwards-Jones formula
- Moments method

- Andreief formula
- Orthogonal Polynomials
- Coulomb gas

Edwards-Jones formula (1976)

$$\mathcal{P}(H_{11},\ldots,H_{NN}) \Rightarrow \rho_N(\lambda) = \left\langle \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\right\rangle$$

(typically $N \to \infty$)

The eigenvalue spectrum of a large symmetric random matrix

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§ Department of Mathematical Physics, University of Birmingham, Birmingham B15 2TT, UK

Received 21 May 1976

Abstract. A new and straightforward method is presented for calculating the eigenvalue spectrum of a large symmetric square matrix each of whose upper triangular elements is described by a Gaussian probability density function with the same mean and variance. Using the $n \to 0$ method, we derive the semicircular eigenvalue spectrum when the mean of each element is zero and show that there is a critical finite mean value above which a single eigenvalue splits off from the semicircular continuum of eigenvalues.

PHYSICAL REVIEW B VOLUME 37, NUMBER 7 1 MARCH 1988

Density of states of a sparse random matrix

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Schlumberger-Doll Research, Old Quarry Road, Ridgefield, Connecticut 06877-4108
(Received 28 April 1987)

The density of states $\rho(\mu)$ of an $N\times N$ real, symmetric, random matrix with elements $0,\pm1$ is calculated in the limit $N\rightarrow\infty$ as a function of the average "connectivity" p, i.e., of the mean number of nonzero elements per row. For $p\rightarrow\infty$, the Wigner semicircular distribution is recovered. For finite p the distribution has tails extending beyond the semicircle, with $\rho(\mu)\sim(ep/\mu^2)^{\mu^2}$ for $\mu^2\rightarrow\infty$. Applications to the theory of "Griffiths singularities" in dilute magnets are discussed.

Tomorrow....

$$\rho_N(\lambda) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle$$

Next, we use the following Sokhotski-Plemelj identity. Let f be a complex-valued function which is defined and continuous on the real line, and let a and b be real constants with $a < 0 < b$. Then:

$$\lim_{\varepsilon \to 0^+} \int_a^b \frac{f(x)}{x \pm \mathrm{i}\varepsilon} dx = \mp \mathrm{i}\pi f(0) + \Pr \int_a^b \frac{f(x)}{x} dx \tag{34}$$

where Pr denotes the Cauchy principal value. A shorthand notation for this theorem is:

$$\frac{1}{x \pm \mathrm{i}\varepsilon} \to \Pr\left(\frac{1}{x}\right) \mp \mathrm{i}\pi\delta(x) \tag{35}$$

We can convert a delta function into a rational function using the Sokhotski-Plemelj identity

$$\rho_N(\lambda) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle$$

$$\rho_N(\lambda) = \frac{1}{\pi N} \lim_{\varepsilon \to 0} \mathrm{Im} \left\langle \sum_{i=1}^{N} \frac{1}{\lambda - \mathrm{i}\varepsilon - \lambda_i} \right\rangle$$

Next, we can convert a rational function into a logarithm

$$\rho_N(\lambda) = \frac{1}{\pi N} \lim_{\varepsilon \to 0} \mathrm{Im} \frac{\partial}{\partial \lambda} \left\langle \sum_{i=1}^{N} \ln(\lambda - \mathrm{i}\varepsilon - \lambda_i) \right\rangle$$

$$\rho_N(\lambda) = \frac{1}{\pi N} \lim_{\varepsilon \to 0} \mathrm{Im} \frac{\partial}{\partial \lambda} \left\langle \sum_{i=1}^{N} \ln(\lambda - \mathrm{i}\varepsilon - \lambda_i) \right\rangle$$

We have a "trace of log" which can be converted into a "log of det"

$$\sum_{i=1}^{N} \ln(\lambda - \mathrm{i}\varepsilon - \lambda_i) = \ln \det((\lambda - \mathrm{i}\varepsilon)\mathbf{I}_N - \mathbf{H})$$

Link between eigenvalues and entries!

$$\ln \det \mathbf{A} = -2 \ln (\det \mathbf{A})^{-1/2}$$

brilliant!

The determinant to the power -1/2 can be traded for a Gaussian integral!

$$\rho_N(\lambda) = \frac{-2}{\pi N} \lim_{\varepsilon \to 0} \mathrm{Im} \frac{\partial}{\partial \lambda} \left\langle \ln \left[\det((\lambda - \mathrm{i}\varepsilon)\mathbf{I}_N - \mathbf{H}) \right]^{-1/2} \right\rangle$$

where now we can use

$$[\det \mathbf{A}]^{-1/2} = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} \prod_{j=1}^{N} dx_j \exp \left(-\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j \right)$$

The log of an integral is a bit inconvenient...

$$\ln z = \lim_{n \to 0} \frac{z^n - 1}{n}$$

Replica Trick

$$\left(\frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} \prod_{j=1}^{N} dx_j \exp\left(-\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j \right) \right)^n =$$

$$\frac{1}{(2\pi)^{Nn/2}} \int_{-\infty}^{\infty} \prod_{j=1}^{N} \prod_{a=1}^{n} dx_{ja} \exp\left(-\frac{1}{2} \sum_{i,j,a} x_{ia} A_{ij} x_{ja} \right)$$

n copies of the original integral ...

In summary ...

$$\rho_N(\lambda) = \frac{-2}{\pi N} \lim_{\varepsilon\to 0} \mathrm{Im} \frac{\partial}{\partial \lambda} \left[\lim_{n\to 0} \left(\frac{\mathcal{I}_\varepsilon(n,\lambda) - 1}{n} \right) \right]$$

where ...

$$\mathcal{I}_\varepsilon(n,\lambda) := \frac{1}{(2\pi)^{Nn/2}} \int \prod_{i,j=1}^{N} dH_{ij} \underline{\mathcal{P}(H_{11},\ldots,H_{NN})} \int_{-\infty}^{\infty} \prod_{j=1}^{N} \prod_{a=1}^{n} dx_{ja} \exp\left(-\frac{1}{2} \sum_{i,j,a} x_{ia} \left[(\lambda - \mathrm{i}\varepsilon)\delta_{ij} - H_{ij} \right] x_{ja} \right)$$

Typically can be evaluated for $N \to \infty$

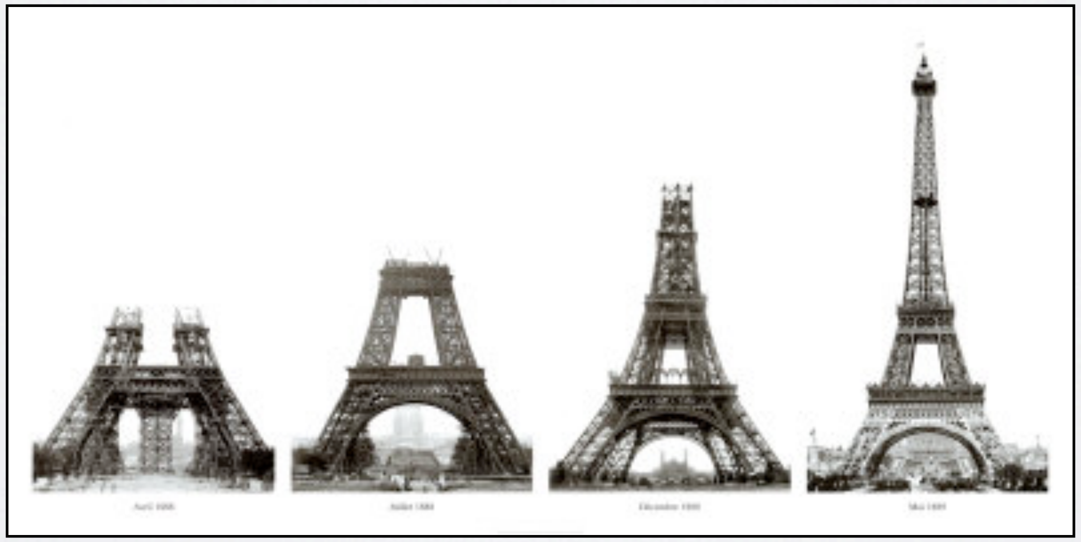
“Shaky” interplay with replica limit...

SUMMARY

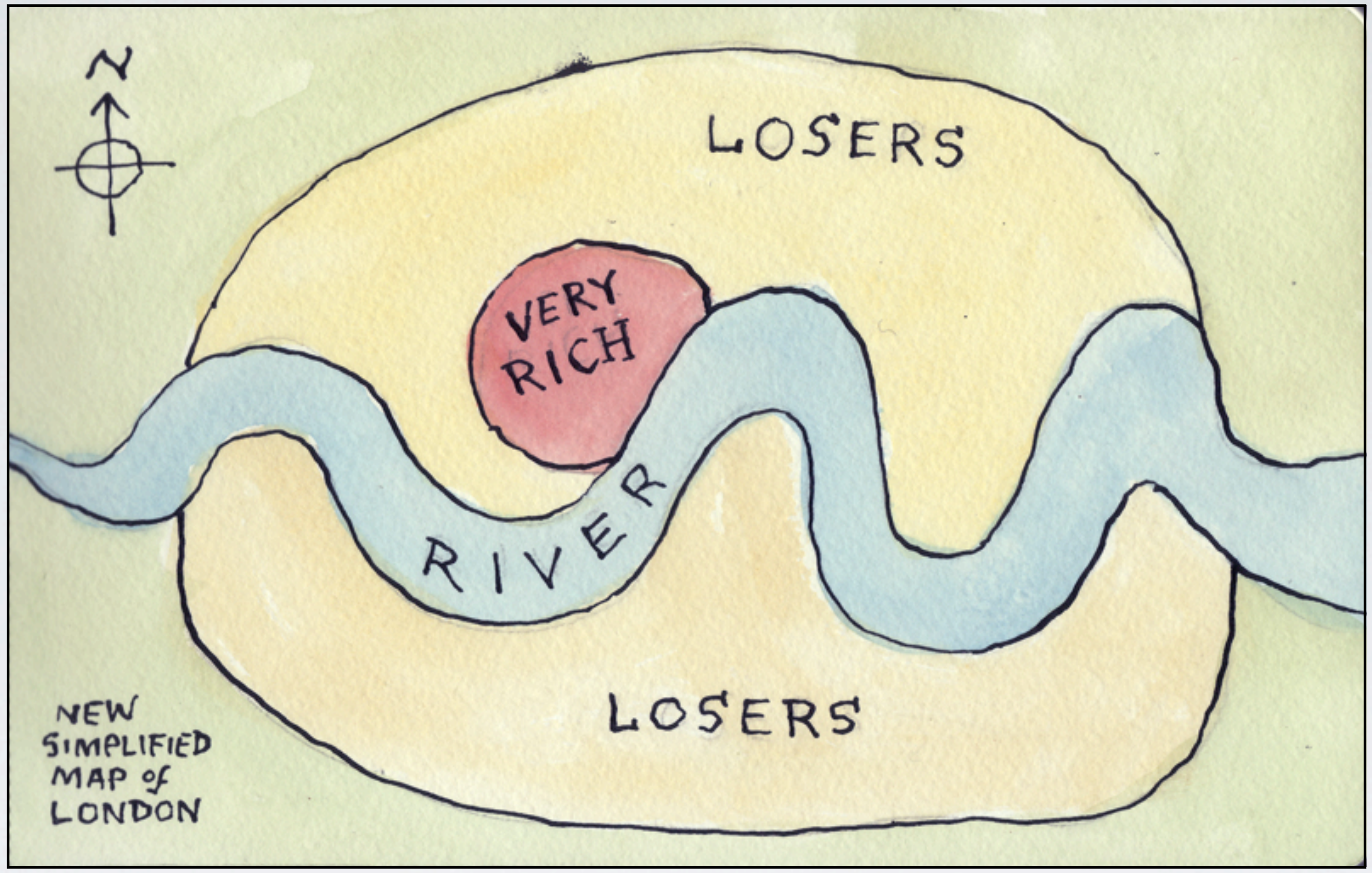
- Eigenvalues of random matrices: **strongly correlated**
- Real spectrum: independent entries or rotational invariance
- Many more analytical tools for invariant models
- "Semicircle" law (quite robust) and level repulsion (quite universal)
- Modern applications (Riemann zeta, non-intersecting Brownian paths, finance, telecommunications....)
- Edwards-Jones formula for the average density of states

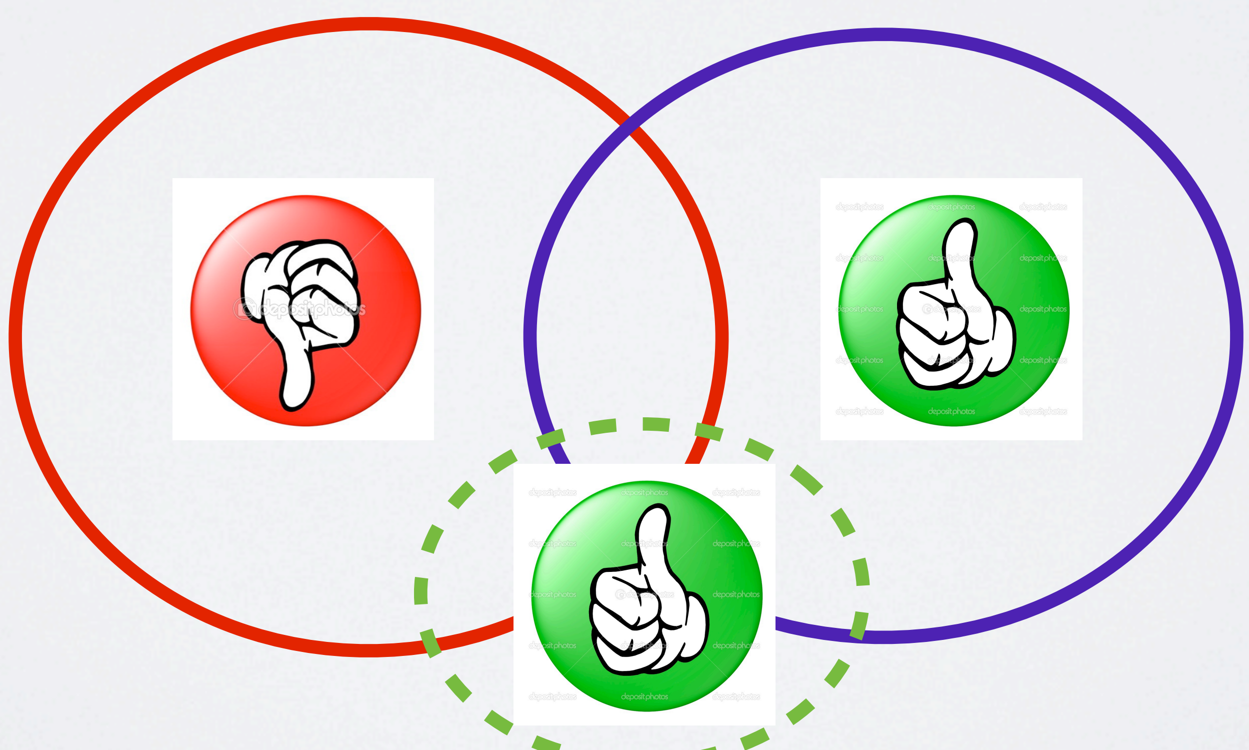
RANDOM MATRIX THEORY AND PRACTICE: OLD TRICKS FOR NEW DOGS

Pierpaolo Vivo
(LPTMS - CNRS - Paris XI)

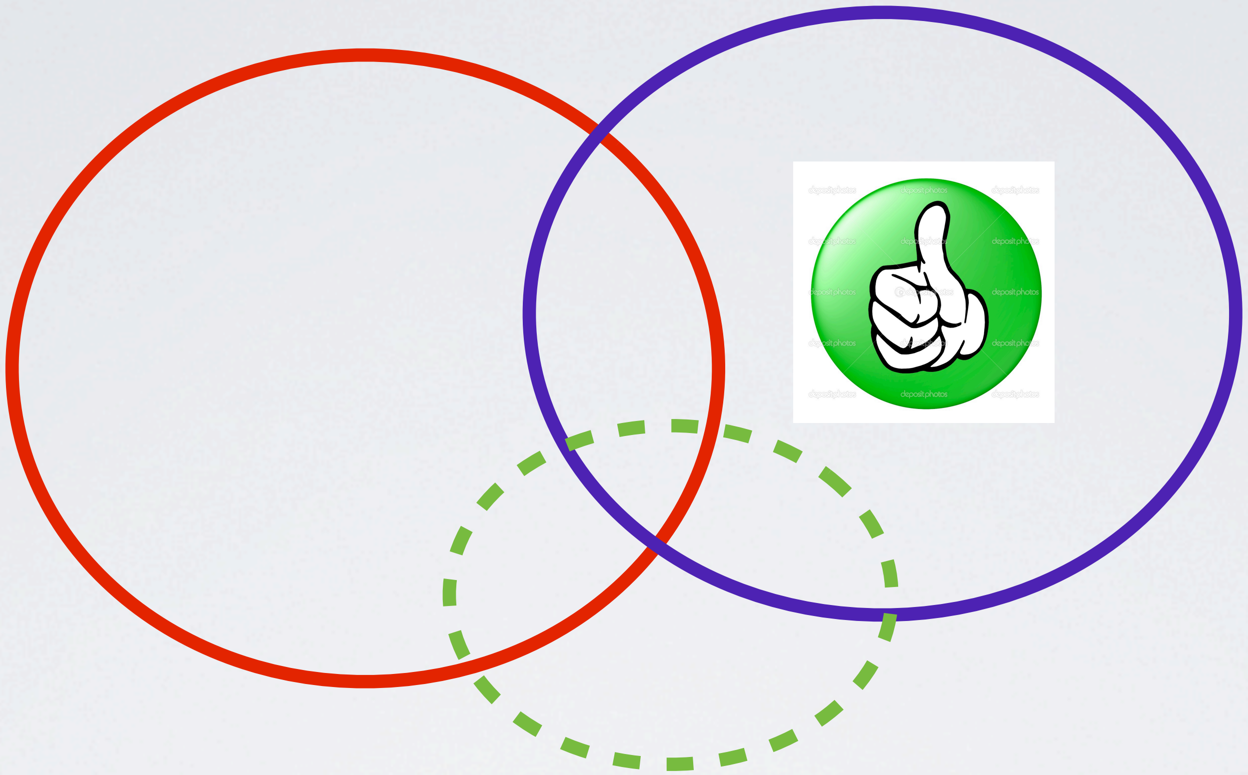


Simplified summary





$$\mathcal{P}(H_{11},\ldots,H_{NN}) \longrightarrow \mathcal{P}(\lambda_1,\ldots,\lambda_N)$$



$$\mathcal{P}[\mathbf{H}] = \phi\left(\mathrm{Tr}\mathbf{H},\ldots,\mathrm{Tr}\mathbf{H}^N\right)$$

Weyl's lemma

$$\mathcal{P}(\lambda_1,\ldots,\lambda_N) = C_{N,\beta}\,\phi\left(\sum_{i=1}^{N}\lambda_i,\ldots,\sum_{i=1}^{N}\lambda_i^N\right)\prod_{j<k}|\lambda_j-\lambda_k|^\beta$$

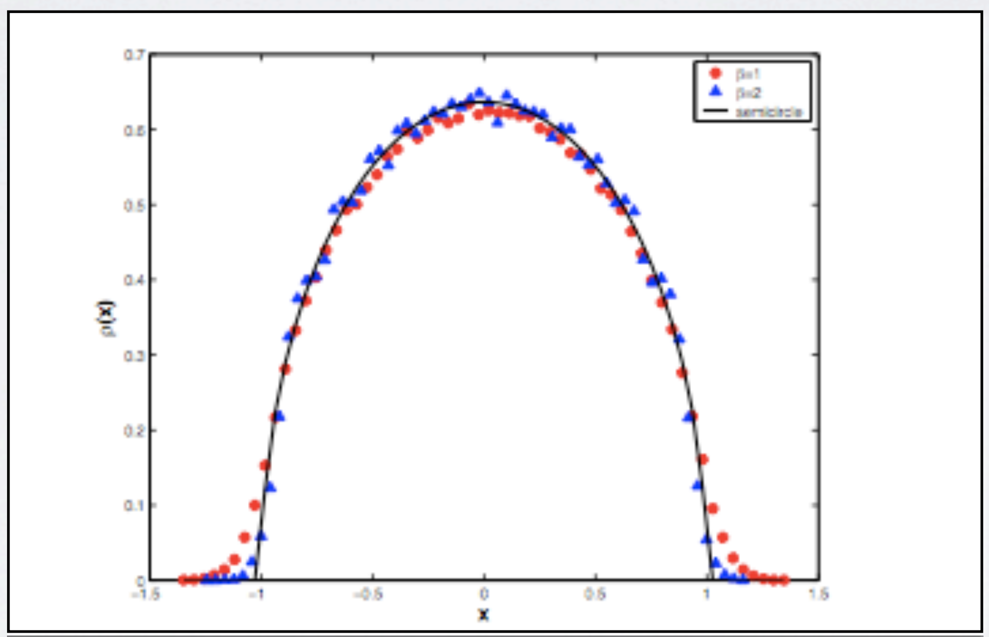
Rotationally Invariant Models

$$\mathcal{P}_\beta(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_{N,\beta}} e^{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

$\beta = 1, 2, 4$

Confinement
(non-universal)

Level Repulsion
(universal)



Strongly Correlated Random Variables!!

Distribution of largest eigenvalue

$$\mathbb{P}_N[\lambda_{\max} < x] = \int^x \cdots \int^x d\lambda_1 \cdots d\lambda_N \mathcal{P}(\lambda_1, \ldots, \lambda_N)$$

Strongly Correlated Random Variables!!

One step back: i.i.d. random variables

$$\{X_1, \ldots, X_N\}$$ i.i.d. sampled from $$p(x)$$

- Law of Large Number (LLN)

$$\overline{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$$

converges to the expected value

$$\overline{X}_n \to \mu \qquad \text{for} \qquad n \to \infty$$

where X_1, X_2, ... is an infinite sequence of i.i.d. integrable random variables with expected value $E(X_1) = E(X_2) = ... = \mu$.

- Central Limit Theorem (CLT)

the law of large numbers, $S_n/n \to \mu$.[14] If in addition each X_i has finite variance σ^2, then by the central limit theorem,

$$\frac{S_n - n\mu}{\sqrt{n}} \to \xi,$$

where ξ is distributed as $N(0, \sigma^2)$. This provides values of the first two constants in the informal expansion

$$S_n \approx \mu n + \xi\sqrt{n}.$$

One step back: i.i.d. random variables

$$\{X_1, \ldots, X_N\}$$ i.i.d. sampled from $$p(x)$$

- Law of Large Number (LLN)
- Central Limit Theorem (CLT)

⬇

They concern the **sum**

What about the maximum?

Extreme Value Theory

i.i.d. random variables: the threefold way for the maximum

$$X_{\max} = \max_i \{X_i\}$$

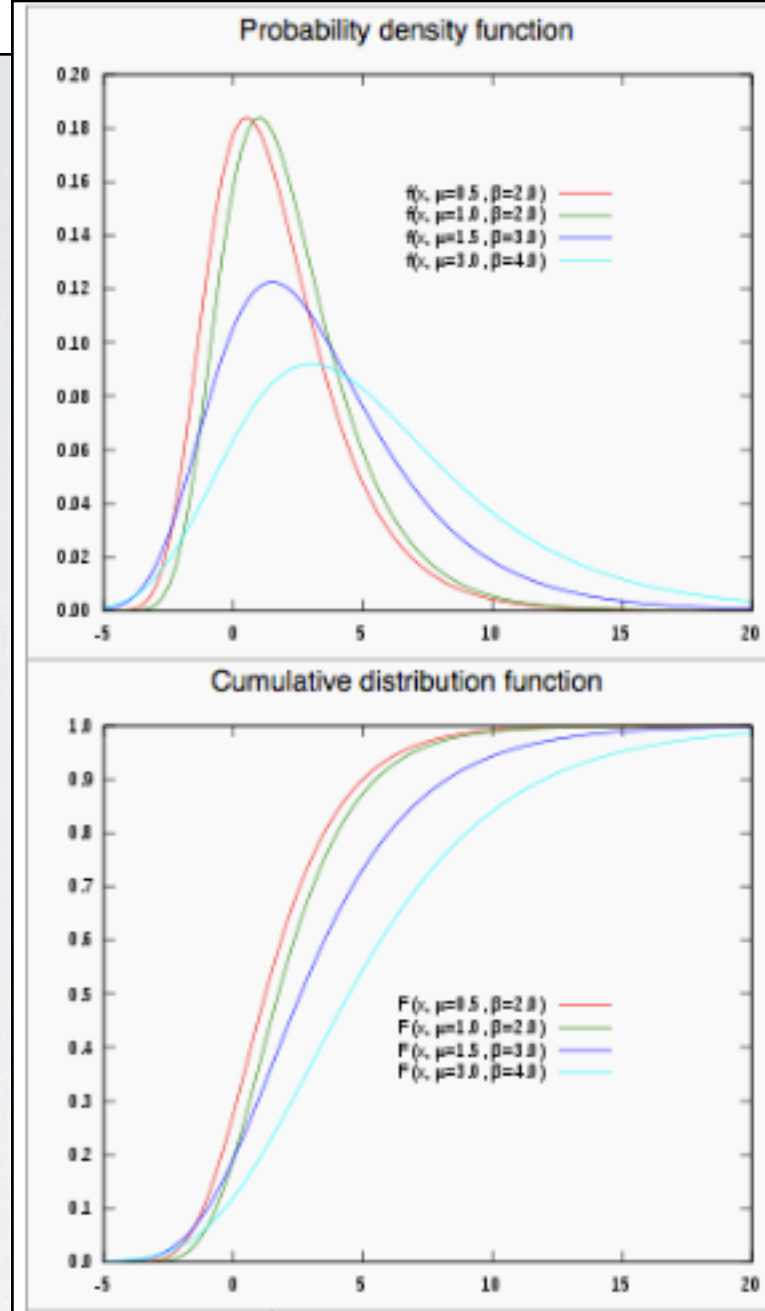
Only 3 universality classes, depending on the tails of $p(x)$

- **Gumbel**: fast decaying
- **Fréchet**: power laws
- **Weibull**: compact support

[Fisher–Tippett–Gnedenko theorem]

$$\lim_{N\to\infty} \mathbb{P}\left[\frac{X_{\max} - a_N}{b_N} \leq x\right] = F_{\mathrm{I}}(x) = \exp(-\exp(-x))$$

Gumbel



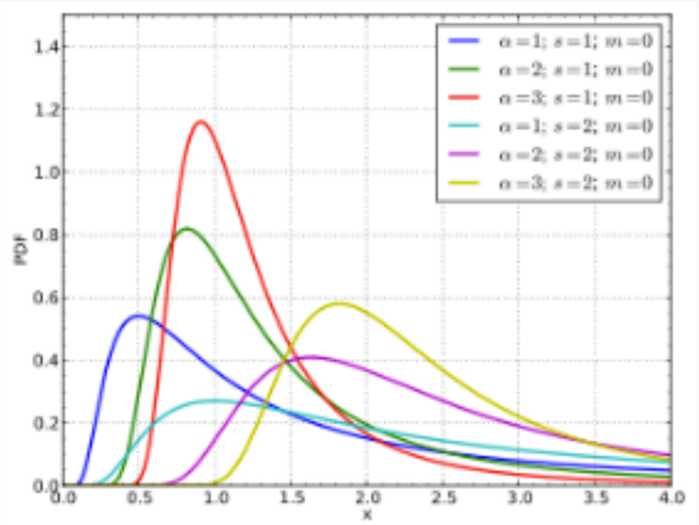
Parameters	μ location (real) $\beta > 0$ scale (real)
Support	$x \in (-\infty; +\infty)$
pdf	$\frac{1}{\beta}e^{-(z+e^{-z})}$ where $z = \frac{x-\mu}{\beta}$
CDF	$e^{-e^{-(x-\mu)/\beta}}$

$$\lim_{N\to\infty} \mathbb{P}\left[\frac{X_{\max} - a_N}{b_N} \le x\right] = F_{\mathrm{II}}(x) = \begin{cases} 0 & x < 0 \\ e^{-1/x^{\gamma}} & x \ge 0 \end{cases}$$

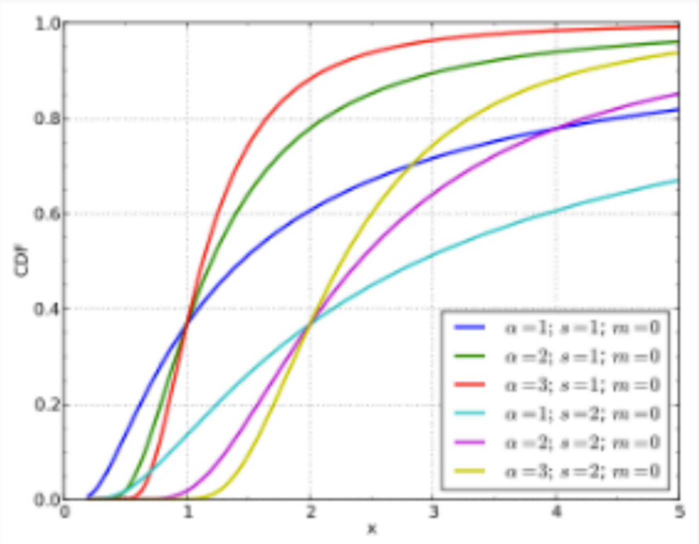
Fréchet

Fréchet

Probability density function



Cumulative distribution function



Parameters	$\alpha \in (0, \infty)$ shape. (Optionally, two more parameters) $s \in (0, \infty)$ scale (default: $s = 1$) $m \in (-\infty, \infty)$ location of minimum (default: $m = 0$)
Support	$x > m$
pdf	$\frac{\alpha}{s} \left(\frac{x-m}{s}\right)^{-1-\alpha} e^{-\left(\frac{x-m}{s}\right)^{-\alpha}}$
CDF	$e^{-\left(\frac{x-m}{s}\right)^{-\alpha}}$

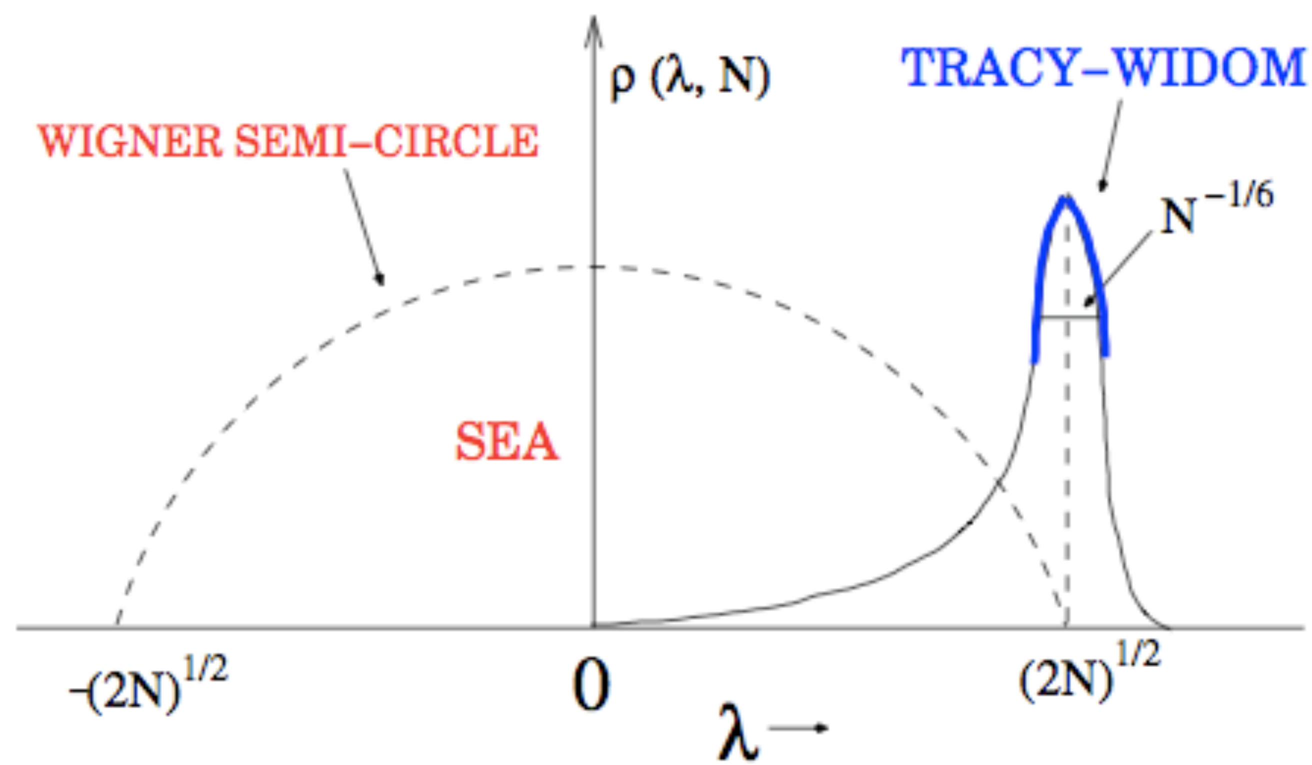
$$\lim_{N\to\infty} \mathbb{P}\left[\frac{X_{\max} - a_N}{b_N} \leq x\right] = F_{\mathrm{III}}(x) = \begin{cases} e^{-|x|^\gamma} & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Weibull

What about
Strongly Correlated Random Variables?

Largest Eigenvalue Gaussian Ensemble

Tracy-Widom distribution for λ_{max}



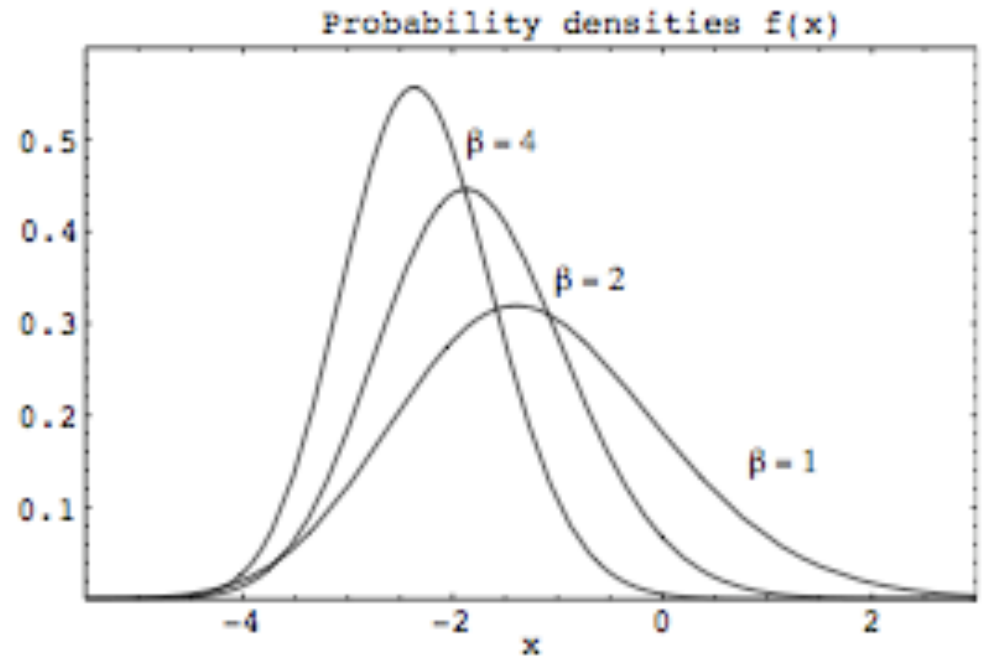
- $\langle \lambda_{max} \rangle = \sqrt{2N}$; typical fluctuation: $|\lambda_{max} - \sqrt{2N}| \sim N^{-1/6}$ (small)
- typical fluctuations are distributed via Tracy-Widom (1994):
- cumulative distribution:

$$\text{Prob}[\lambda_{max} \le t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$$

- Prob. density (pdf): $f_\beta(z) = dF_\beta(z)/dz$
- $F_\beta(z) \to$ obtained from solution of Painlevé-II equation

[borrowed from S.N. Majumdar,
"Top eigenvalue of a random matrix: a tale of tails."]

Tracy-Widom distribution for λ_{max}



- Tracy-Widom density $f_\beta(x)$ depends explicitly on β.
- Asymptotics: $f_\beta(x) \sim \exp\left[-\frac{\beta}{24}|x|^3\right]$ as $x \to -\infty$

$$\sim \exp\left[-\frac{2\beta}{3}\, x^{3/2}\right] \quad \text{as} \quad x \to \infty$$

Applications: Growth models, Directed polymer, Sequence Matching .....
(Baik, Deift, Johansson, Prahofer, Spohn, Johnstone,....)

A recent 'simpler' derivation of Tracy-Widom for $\beta = 2 \rightarrow$ [Nadal and Majumdar 2011]

Tracy-Widom distribution

$$\lambda_{\max} \approx \sqrt{2N} + a_\beta N^{-1/6} \chi$$

$$\mathcal{P}(\chi \leq x) = F_\beta(x)$$

$$F_2(x) = \exp\left[-\int_x^\infty (z-x) q^2(z) dz\right]$$

$$q'' = 2q^3 + zq$$

Painlevé II

Painlevé transcendents and their appearance in physics

$$y'' = F(x, y, y')$$

Rational function of its arguments — **E. Picard** (1889)

All **movable singularities** are restricted to **poles**
(no movable branch points)

50 → **44**

(a) **linear 2nd order DEs**
(b) **Weierstrass DE**
(c) **Riccati DE**

$$(y')^2 = 4y^3 - g_2 y - g_3$$

$$y' = a(x)\,y^2 + b(x)\,y + c(x)$$

[**6** **Painlevé equations P_I – P_{VI}** nonlinear special functions]

P. Painlevé (1900,1902)
B. Gambier (1905)
R. Fuchs (1910)

Movable singularity

From Wikipedia, the free encyclopedia

This article includes a list of references, related reading or external links, but **its sources remain unclear because it lacks inline citations.** P more precise citations. *(January 2011)*

In the theory of ordinary differential equations, a **movable singularity** is a point where the solution of the equation behaves badly and which is "movable" in the sense that its location depends on the initial conditions of the differential equation.[1] Suppose we have an ordinary differential equation in the complex domain. Any given solution $y(x)$ of this equation may well have singularities at various points (i.e. points at which it is not a regular holomorphic function, such as branch points, essential singularities or poles). A singular point is said to be **movable** if its location depends on the particular solution we have chosen, rather than being fixed by the equation itself.

For example the equation

$$\frac{dy}{dx} = \frac{1}{2y}$$

has solution $y = \sqrt{x - c}$ for any constant c. This solution has a branchpoint at $x = c$, and so the equation has a movable branchpoint (since it depends on the choice of the solution, i.e. the choice of the constant c).

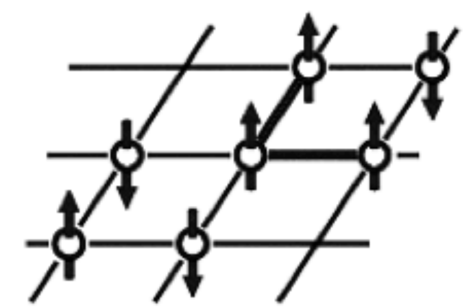
It is a basic feature of linear ordinary differential equations that singularities of solutions occur only at singularities of the equation, and so linear equations do not have movable singularities.

When attempting to look for 'good' nonlinear differential equations it is this property of linear equations that one would like to see: asking for no movable singularities is often too stringent, instead one often asks for the so-called Painlevé property: 'any movable singularity should be a pole', first used by Sofia Kovalevskaya.

Painlevé transcendents and their appearance in physics

- **2D Ising model**

$$H_{\text{int}}^{(2D)} = -J \sum_{j,k} \left(\boldsymbol{\sigma}_{j,k}\, \boldsymbol{\sigma}_{j,k+1} + \boldsymbol{\sigma}_{j,k}\, \boldsymbol{\sigma}_{j+1,k}\right)$$



Existence of the 2nd order phase transition, Critical temperature, Spontaneous magnetisation

T. Wu, B. McCoy, C. Tracy, and E. Barouch (1976)

$$\langle \boldsymbol{\sigma}_{00} \boldsymbol{\sigma}_{MN} \rangle \Big|_{\substack{T \to T_c^{\pm} \\ R=(M^2+N^2)^{1/2} \to \infty}} = \boldsymbol{F}^{\pm} \left([\sigma_{\text{III}}];\ r = \frac{R}{\xi(T)} \right)$$

$$\xi(T) = \frac{(T_c/4J)}{|1 - T/T_c|}$$

$$\tanh(J/T_c) = \sqrt{2} - 1$$

σP_{III}

‣ Painlevé transcendents and their appearance in physics

- **Impenetrable Bose gas** $g \to \infty$

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + g \sum_{i<j} \delta(z_i - z_j)$$

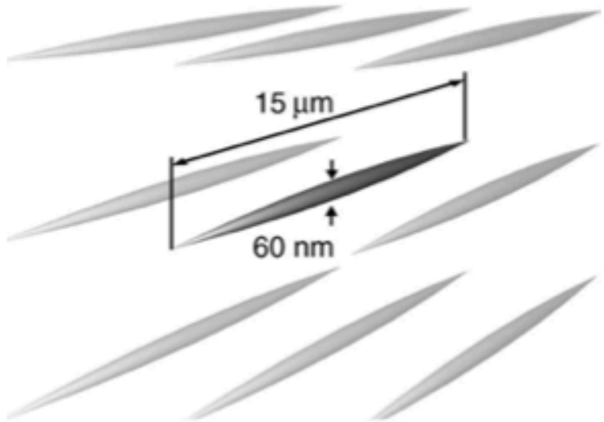


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FIG. 1. The geometry and size of trapped 1D gases in a two-dimensional optical lattice. The spacing between the 1D tubes in the horizontal and vertical direction is 413 nm.

M. Jimbo, T. Miwa, Y. Môri, and M. Sato (1980)

$$\varrho_N(x) = N \int_0^L dz_2 \cdots dz_N \, \Psi^*(x, z_2, \cdots, z_N) \, \Psi(0, z_2, \cdots, z_N)$$

$$\varrho_\infty(x) = \lim_{N\to\infty} \varrho_N(x)\Big|_{L=N} = \exp\left(\int_0^{\pi x} \frac{dt}{t} \sigma_V(t)\right)$$

⬅ σP_V

Commun. Math. Phys. 159, 151–174 (1994)

Communications in
Mathematical
Physics

Level-Spacing Distributions and the Airy Kernel

Craig A. Tracy[1,⋆], **Harold Widom**[2,⋆⋆]

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Received: 1 December 1992/in revised form: 24 March 1993

Instanton Induced Large N Phase Transitions in Two and Four Dimensional QCD

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The double scaling limit...

to be $C = D = 0$.

Therefore, $f_1(x)$ obeys the Painleve II equation

$$f_1'' - 4xf_1 - \frac{\pi^2}{2} f_1^3 = 0. \tag{5.15}$$

Now we are in a position to evaluate the free energy in the double scaling limit.

It is a well-known result that U(N) lattice QCD in two dimensions with Wilson's action [54] exhibits a third order phase transition in the large N limit [23, 53]. This is shown by forming the partition function for the plaquettes, which factorizes as a product of partition functions for each individual plaquette. The latter is identified with a zero-dimensional unitary matrix model having partition function given by

$$G_N(b) := \left\langle e^{bN\,\mathrm{Tr}(U+U^\dagger)} \right\rangle_{U\in \mathrm{U}(N)}, \tag{1}$$

where the matrices $U \in \mathrm{U}(N)$ are chosen with Haar measure and b is the scaled coupling.

The matrix integral (1) depends only on the N eigenvalues of U, and in terms of these variables it can be written

$$G_N(b) = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \prod_{l=1}^{N} e^{2bN\cos\theta_l} \prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^2. \tag{2}$$

This can be interpreted as a partition function for a classical gas of charged particles, confined to the unit circle, and repelling via logarithmic pair potential $-(1/2)\log|e^{i\theta} - e^{i\phi}|$ at the inverse temperature $\beta = 2$. The charges are also subject to the extensive one-body potential $bN\cos\theta$. In the form (2) the $N \to \infty$ limit can be computed with the result [23]

$$\lim_{N\to\infty} \frac{1}{N^2} \log G_N(b) = \begin{cases} b^2, & 0 < b < \frac{1}{2} \\ 2b - \frac{3}{4} - \frac{1}{2}\log 2b, & b > \frac{1}{2}, \end{cases} \tag{3}$$

which is indeed discontinuous in the third derivative at $b = 1/2$.

It is a well-known result that U(N) lattice QCD in two dimensions with Wilson's action [54] exhibits a third order phase transition in the large N limit [23, 53]. This is shown by forming the partition function for the plaquettes, which factorizes as a product of partition functions for each individual plaquette. The latter is identified with a zero-dimensional unitary matrix model having partition function given by

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a well-defined scaling limit. It turns out that if one zooms in the neighbourhood of the critical point $b = 1/2$ and magnifies it by a factor $N^{2/3}$, i.e., one takes the limit $(1/2 - b) \to 0$, $N \to \infty$, but keeping the product $t = 2^{4/3}(1/2 - b)N^{2/3}$ fixed,

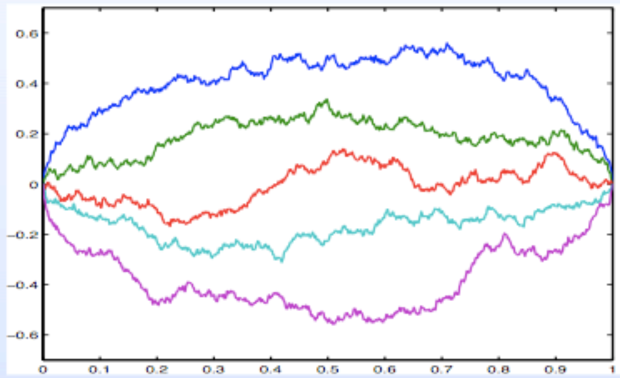
subject to the extensive one-body potential $bN\cos\theta$. In the form (2) the $N \to \infty$ limit can be computed with the result [23]

$$\lim_{N\to\infty} \frac{1}{N^2} \log G_N(b) = \begin{cases} b^2, & 0 < b < \frac{1}{2} \\ 2b - \frac{3}{4} - \frac{1}{2}\log 2b, & b > \frac{1}{2}, \end{cases} \tag{3}$$

which is indeed discontinuous in the third derivative at $b = 1/2$.

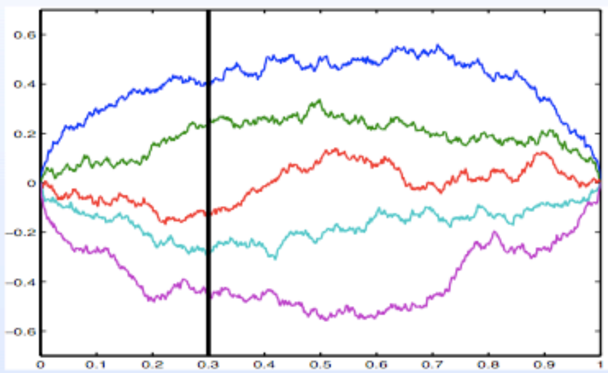
Non-intersecting Brownian motion paths

- Take n independent 1-dimensional Brownian motions with time in $[0, 1]$ conditioned so that:
  - All paths start and end at the same point.
  - The paths **do not intersect** at any intermediate time.



Five non-intersecting Brownian bridges

- **Remarkable fact**: At any intermediate time the positions of the paths have **exactly the same distribution** as the eigenvalues of an $n \times n$ GUE matrix (up to a scaling factor).



Positions of five non-intersecting Brownian paths behave the same as the eigenvalues of a 5×5 GUE matrix

- This interpretation is basic for the connection of random matrix theory with growth models of statistical physics.

Introduction. Since the pioneering work of de Gennes [1], followed up by Fisher [2], the subject of vicious (non-intersecting) random walkers has attracted a lot of interest among physicists. It has been studied in the context of wetting and melting [2], networks of polymers [3] and fibrous structures [1], persistence properties in nonequilibrium systems [4] and stochastic growth models [5, 6]. There also exist connections between the

PHYSICAL REVIEW E VOLUME 52, NUMBER 6 DECEMBER 1995

Vicious walkers and directed polymer networks in general dimensions

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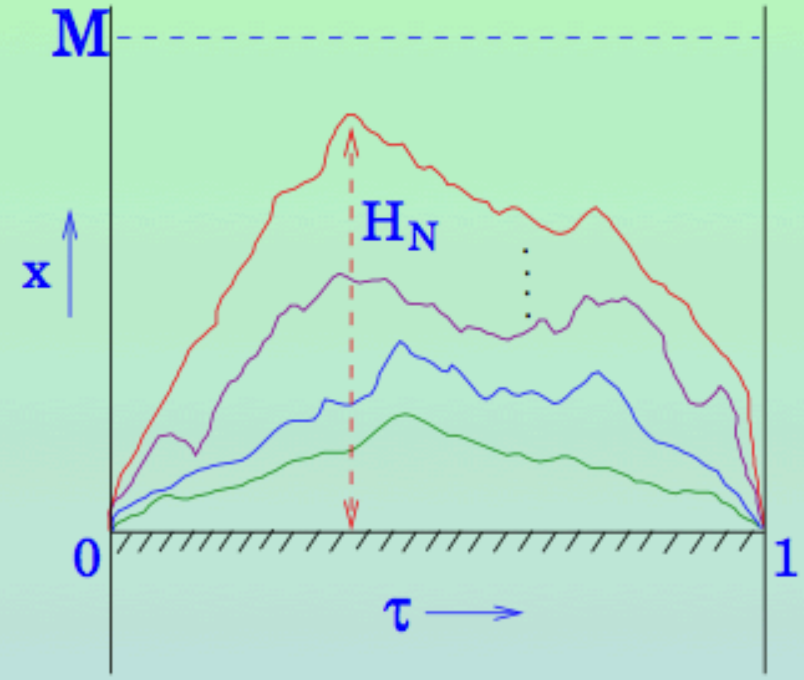
(Received 2 May 1995)

Maximal height of watermelons with a wall

- Cumulative distribution of the maximal height

$$F_N(M) = \Pr[x_N(\tau) \le M,\ \forall 0 \le \tau \le 1] = \int_0^1 d\tau_M \int_0^M dx\, P_N(x, \tau_M)$$



- Path integral for free fermions

[Schehr *et al*. 2008]

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\le j<k\le N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}$$

$N \to \infty$ after centering and scaling, it converges to F_1 (GOE)

Correspondence between YM_2 on the sphere and watermelons

- Partition function of YM_2 on the sphere with gauge group $\mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \mathcal{Z}(A; \mathrm{Sp}(2N))$$

$$\mathcal{Z}(A; \mathrm{Sp}(2N)) = \hat{c}_N\, e^{A(N+\frac{1}{2})\frac{N+1}{12}} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{j=1}^{N} n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N}\sum_{j=1}^{N} n_j^2}$$

- Cumulative distribution of the maximal height of watermelons with a wall

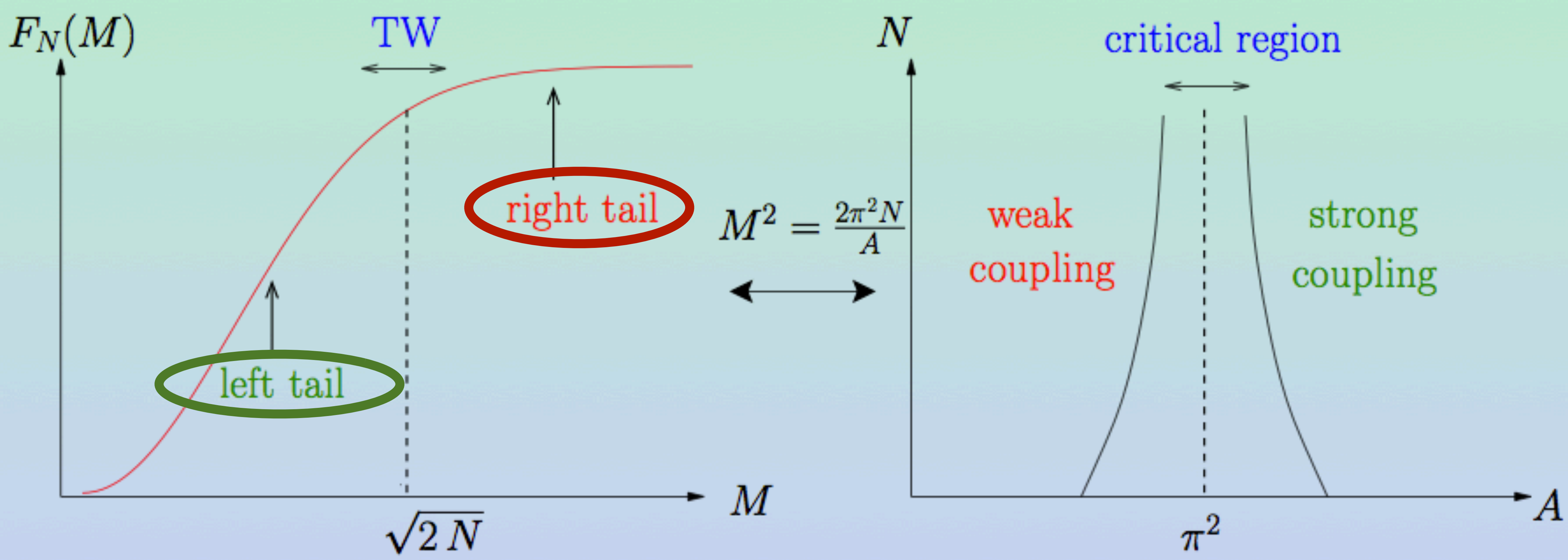
$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \left(\prod_{j=1}^{N} n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{j=1}^{N} n_j^2}$$

$$\propto \mathcal{Z}\left(A = \frac{2\pi^2 N}{M^2}; \mathrm{Sp}(2N)\right)$$

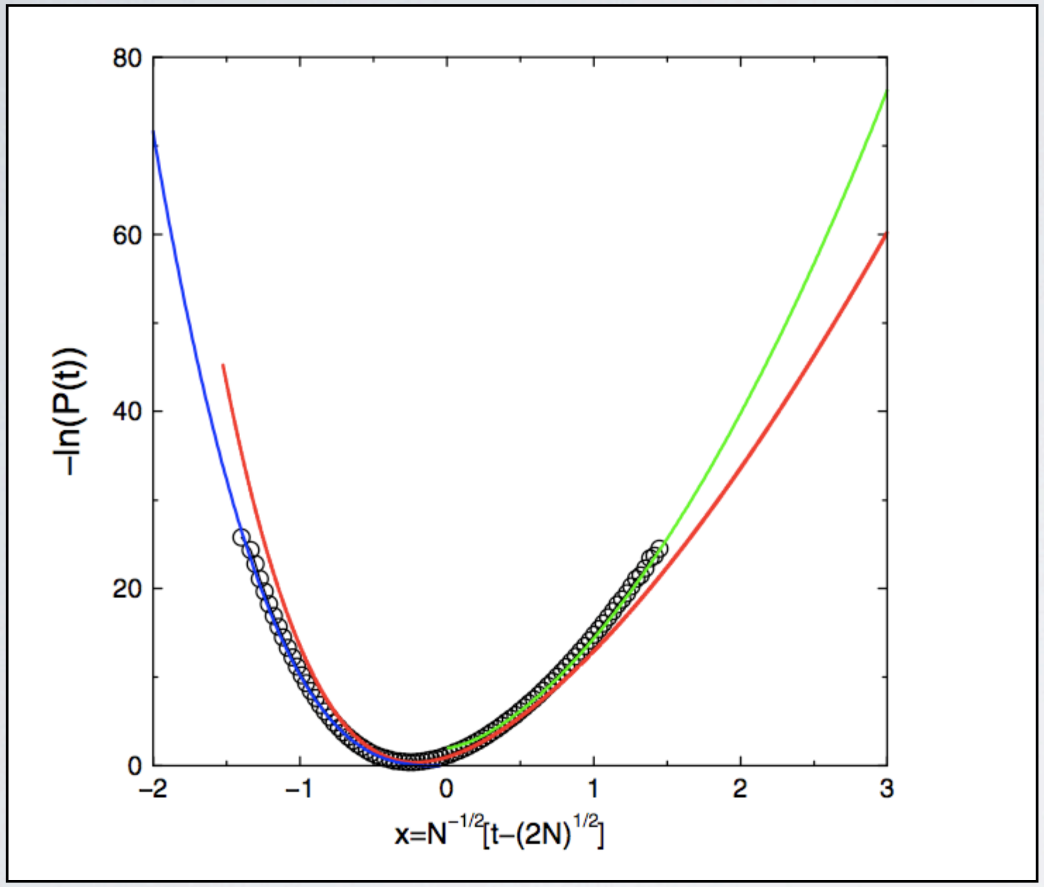
[Forrester *et al.* 2011]

Large N limit of YM_2 and consequences for $F_N(M)$

- Weak-strong coupling transition in YM_2 Durhuus-Olesen '81, Douglas-Kazakov '93



Typical vs. Atypical



PRL **102,** 060601 (2009) PHYSICAL REVIEW LETTERS week ending 13 FEBRUARY 2009

Large Deviations of the Maximum Eigenvalue for Wishart and Gaussian Random Matrices

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Probab. Theory Relat. Fields 120, 1–67 (2001)
Digital Object Identifier (DOI) 10.1007/s004400000115

G. Ben Arous · A. Dembo · A. Guionnet

Aging of spherical spin glasses

$$\mathcal{P}(\lambda_{\max}=t)\approx\begin{cases}\exp\left(-\beta N^2\psi_-\left(\frac{t}{\sqrt{N}}\right)+\dots\right) & \text{for } t<\sqrt{2N} \text{ and } |t-\sqrt{2N}|\approx\mathcal{O}(N)\\ \frac{1}{a_\beta N^{-1/6}}F'_\beta\left(\frac{t-\sqrt{2N}}{a_\beta N^{-1/6}}\right) & \text{for } |t-\sqrt{2N}|\approx\mathcal{O}(N^{-1/6})\\ \exp\left(-\beta N\psi_+\left(\frac{t}{\sqrt{N}}\right)+\dots\right) & \text{for } t>\sqrt{2N} \text{ and } |t-\sqrt{2N}|\approx\mathcal{O}(N)\end{cases}$$

$$\lim_{N\to\infty}\frac{1}{\beta N^2}\ln\mathcal{P}(\lambda_{\max}=z\sqrt{N})=-\psi_-(z)\qquad\text{for } z<\sqrt{2}$$

$$\lim_{N\to\infty}\frac{1}{\beta N}\ln\mathcal{P}(\lambda_{\max}=z\sqrt{N})=-\psi_+(z)\qquad\text{for } z>\sqrt{2}$$

Applications of TW distribution

[Takeuchi and Sano, PRL 2010]

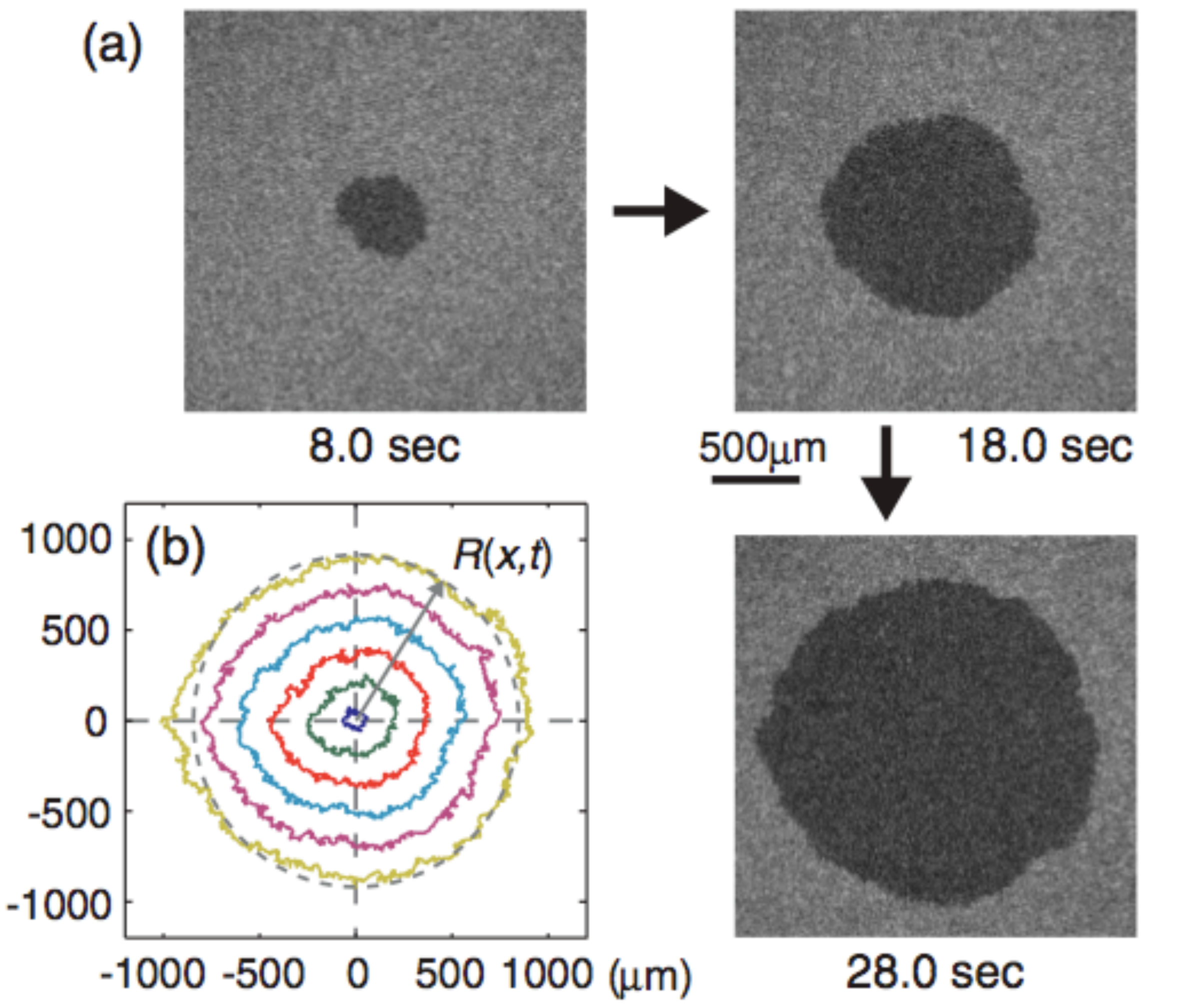
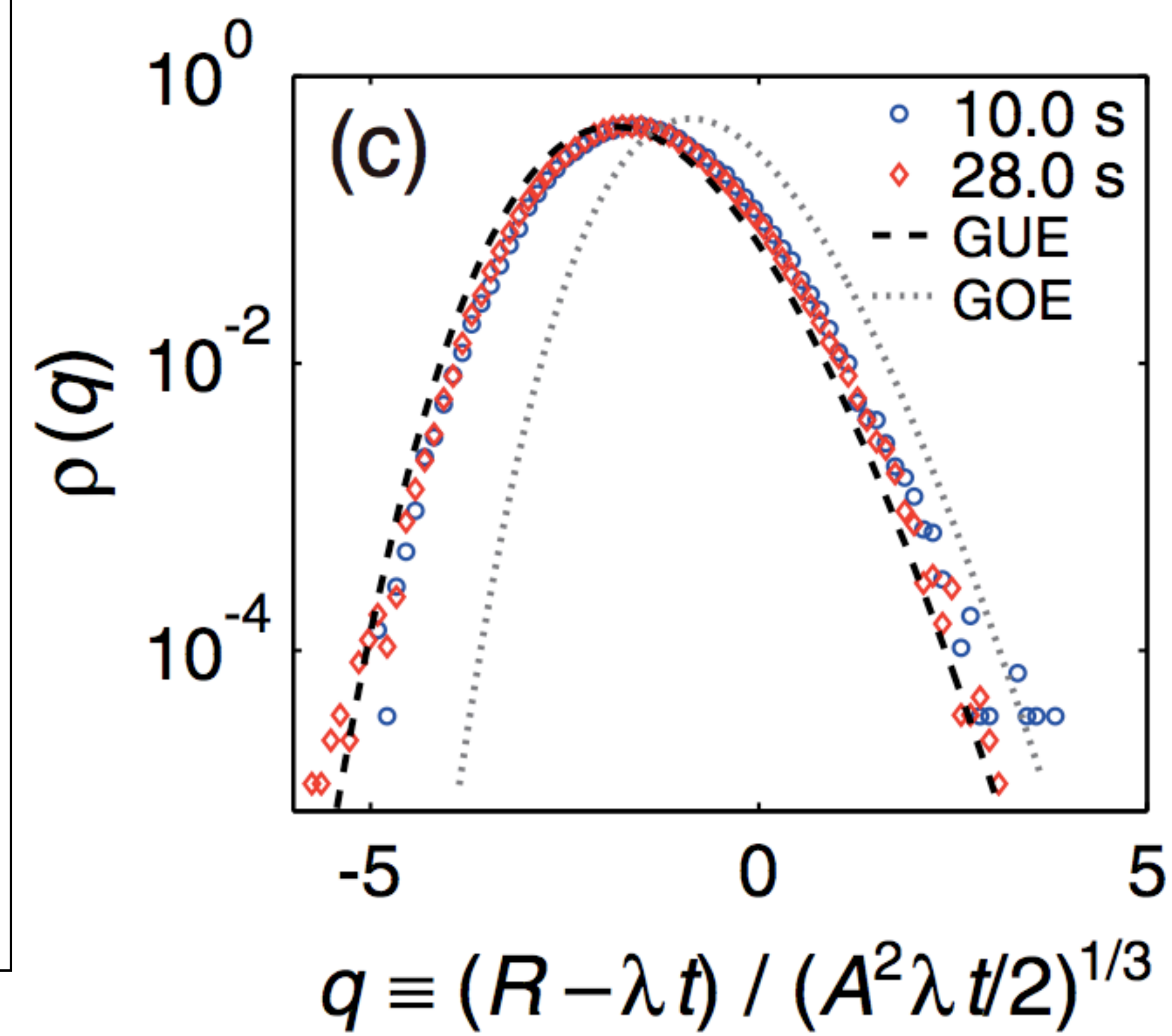


FIG. 1 (color online). Growing DSM2 cluster. (a) Images. Indicated below is the elapsed time after the emission of laser pulses. (b) Snapshots of the interfaces taken every 5 s in the range 2 s $\le t \le$ 27 s. The gray dashed circle shows the mean radius of all the droplets at $t = 27$ s. The coordinate x at this time is defined along this circle.



VOLUME 56, NUMBER 9 PHYSICAL REVIEW LETTERS 3 MARCH 1986

Dynamic Scaling of Growing Interfaces

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and

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(Received 12 November 1985)

A model is proposed for the evolution of the profile of a growin
growth is solved exactly, and exhibits nontrivial relaxation patterns.
ied by dynamic renormalization-group techniques and by mappings
random directed-polymer problem. The exact dynamic scaling form
interface is in excellent agreement with previous numerical simulati
more dimensions.

The interface profile, suitably coarse-grained, is described by a height $h(\mathbf{x},t)$. As usual, it is convenient to ignore overhangs so that h is a single-valued function of $\mathbf{x}$. The simplest nonlinear Langevin equation for a local growth of the profile is given by[12]

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{x},t). \tag{1}$$

The first term on the right-hand side describes relaxation of the interface by a surface tension ν. The second term is the lowest-order nonlinear term that can appear in the interface growth equation, and is justified later on with the Eden model as an example. Edwards and Wilkinson[13] have studied Eq. (1) without the nonlinear term, but we demonstrate that such a term is necessary, and responsible for the unusual properties of the growing interface. Higher-order terms can also be present, but they are irrelevant, and will not modify the universal scaling properties. The noise $\eta(\mathbf{x},t)$ has a Gaussian distribution with $\langle\eta(\mathbf{x},t)\rangle = 0$, and

$$\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = 2D\,\delta^d(\mathbf{x}-\mathbf{x}')\,\delta(t-t'),$$

PRL **104,** 230602 (2010) PHYSICAL REVIEW LETTERS week ending 11 JUNE 2010

One-Dimensional Kardar-Parisi-Zhang Equation: An Exact Solution and its Universality

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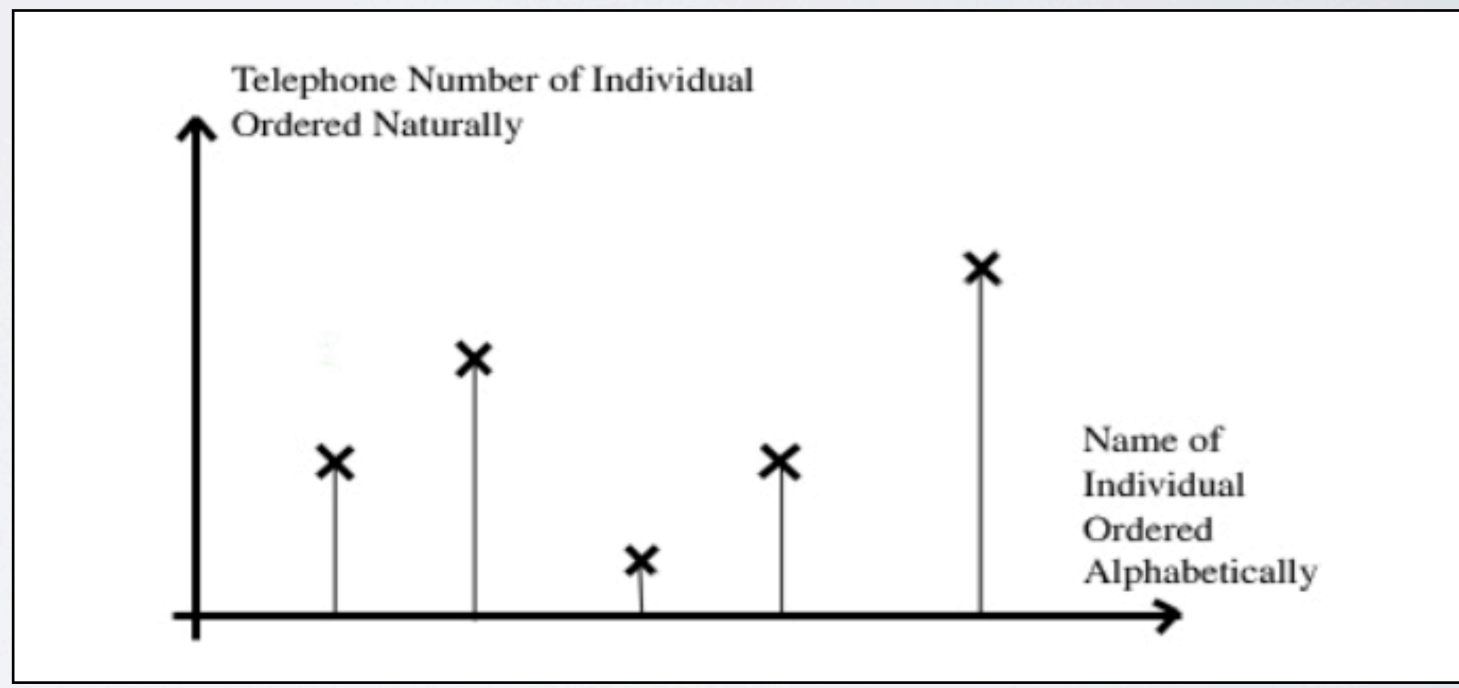
(Received 15 February 2010; revised manuscript received 10 May 2010; published 11 June 2010)

We report on the first exact solution of the Kardar-Parisi-Zhang (KPZ) equation in one dimension, with an initial condition which physically corresponds to the motion of a macroscopically curved height profile. The solution provides a determinantal formula for the probability distribution function of the height $h(x, t)$ for all $t > 0$. In particular, we show that for large t, on the scale $t^{1/3}$, the statistics is given by the Tracy-Widom distribution, known already from the Gaussian unitary ensemble of random matrix theory. Our solution confirms that the KPZ equation describes the interface motion in the regime of weak driving force. Within this regime the KPZ equation details how the long time asymptotics is approached.

“Are Tracy and Widom in Your Local Telephone Directory?”

Ryan Witko
Advisor: Percy Deift



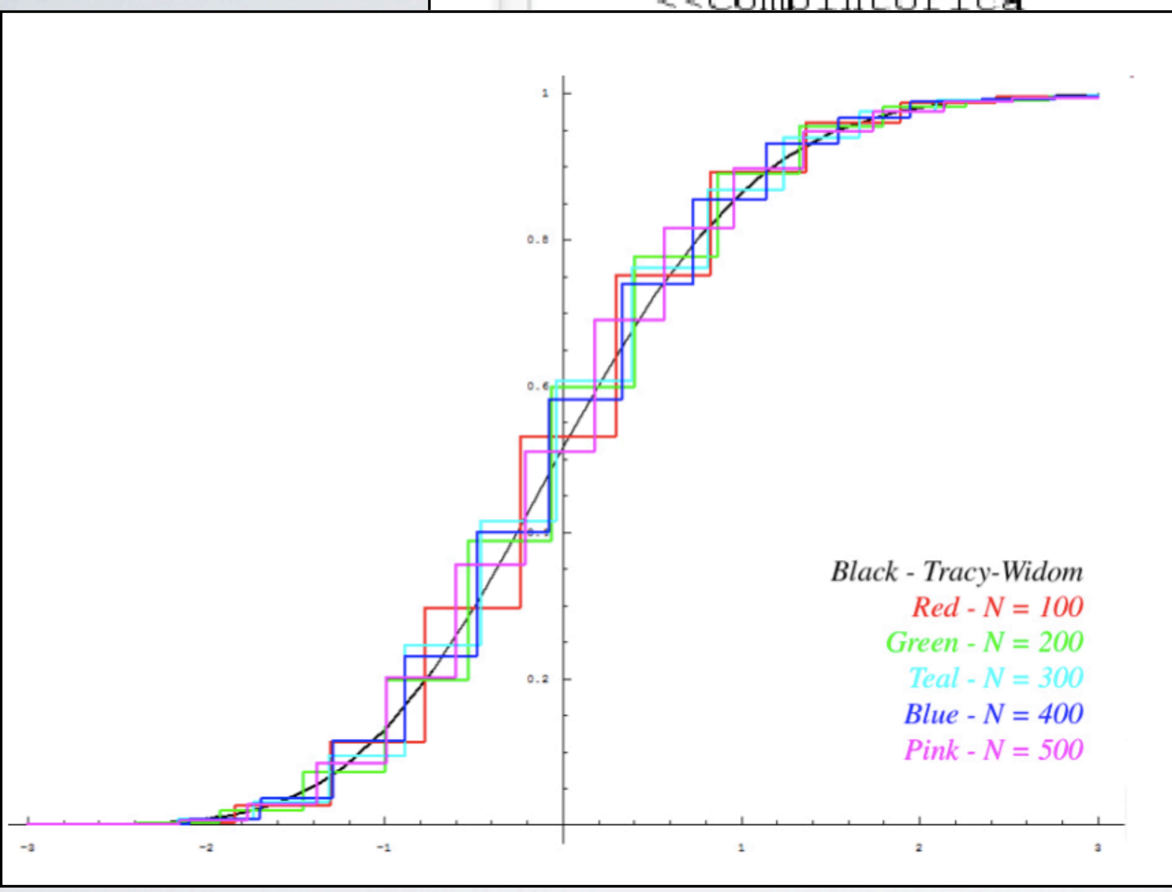
Definition:

The longest increasing (contiguous) subsequence of a given sequence is the subsequence of increasing terms containing the largest number of elements. For example, the longest increasing subsequence of the permutation {6, 3, 4, 8, 10, 5, 7, 1, 9, 2} is {3, 4, 8, 10}.

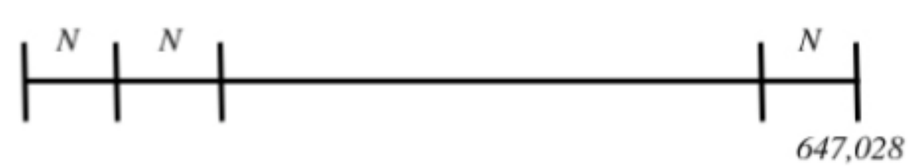
It can be coded in *Mathematica* as follows.

```
<<Combintorica`
creasingSubsequence[p_] :=
], Length[#1] >= Length[#2]&]
```

More information »



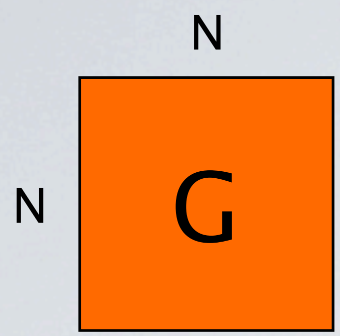
We broke the 647,028 entries into successive samples each containing *N* entries.

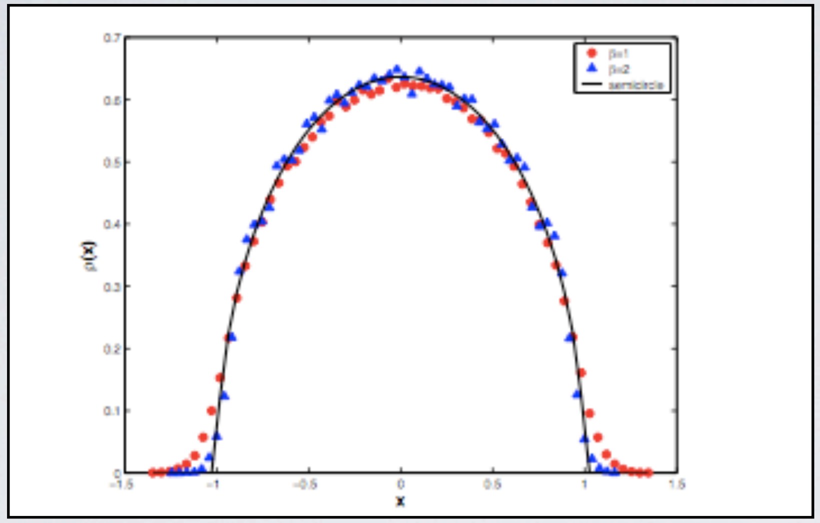


Jinho Baik, Kurt Johansson and Percy Deift showed that as $N\to\infty$

(3)
$$Prob\left(\frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \le t\right) \to F(t)$$

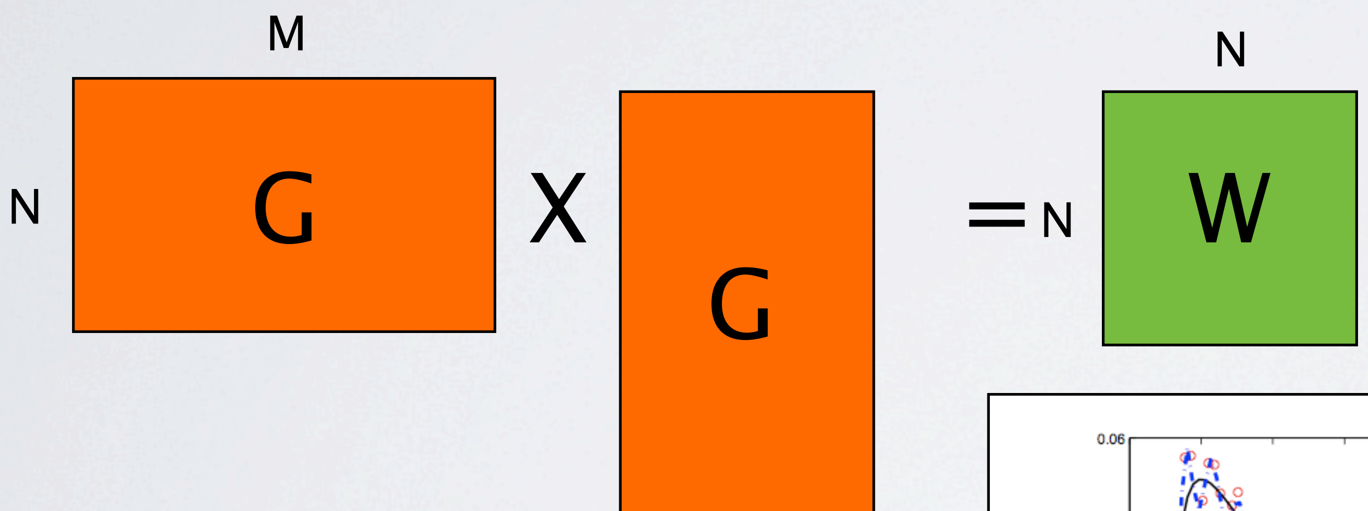
The function *F(t)* was shown by Craig Tracy and Harold Widom to be the distribution of the largest eigenvalue of a random matrix in the Gaussian Unitary Ensemble (GUE). It





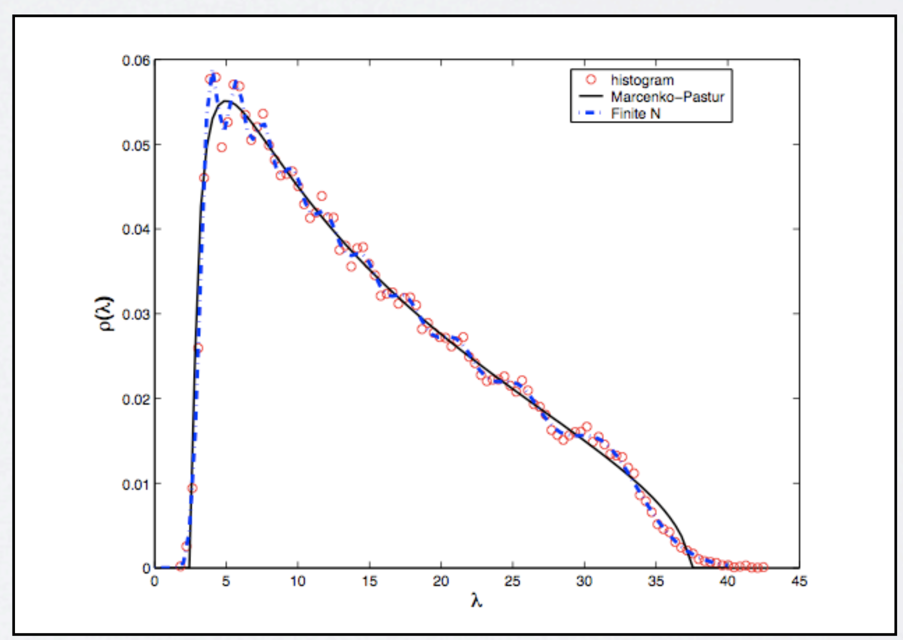
Gaussian

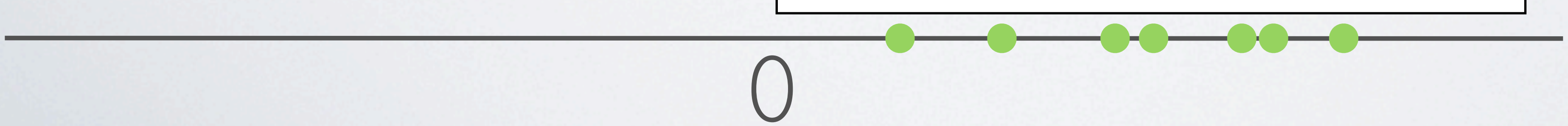
$$\mathbf{G}_{ij} \sim \mathcal{N}(0,1)$$



Wishart

$$\mathbf{W} = \mathbf{G}\mathbf{G}^{\dagger}$$





RAPID COMMUNICATIONS

PHYSICAL REVIEW E **85**, 020101(R) (2012)

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

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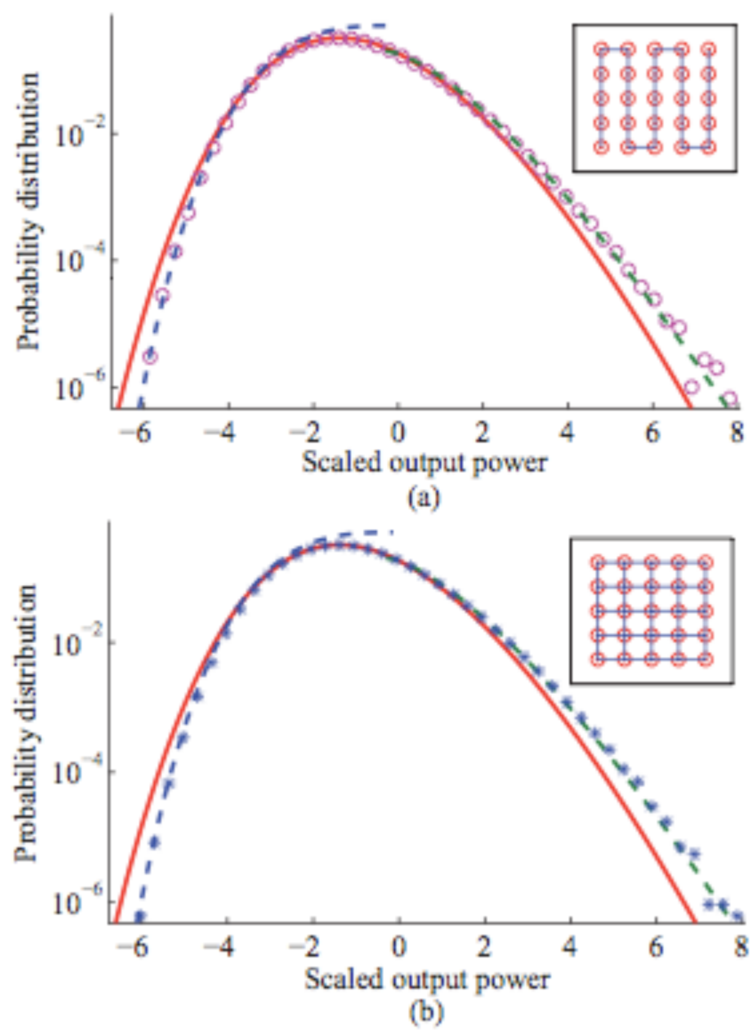
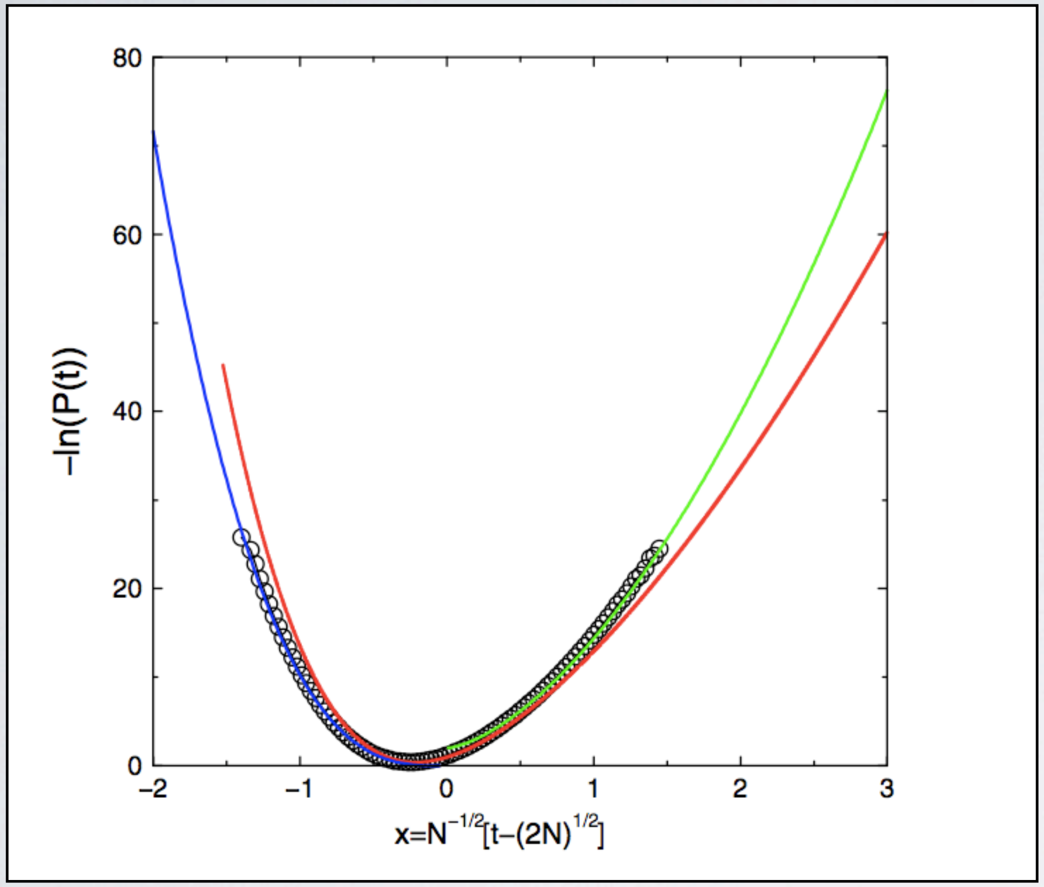
(Received 16 December 2011; published 1 February 2012)



Recently, Majumdar and Vergassola (MV) calculated the probability of large deviations of the maximal eigenvalue [12–14] above the mean and Pierpaolo, Majumdar, and Bohigas (PMB) calculated below the mean. The MV and the PMB distributions were numerically confirmed, but so far eluded experimental demonstration.

Typical vs. Atypical



PRL **102,** 060601 (2009) PHYSICAL REVIEW LETTERS week ending 13 FEBRUARY 2009

Large Deviations of the Maximum Eigenvalue for Wishart and Gaussian Random Matrices

Satya N. Majumdar[1] and Massimo Vergassola[2]
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(Received 14 November 2008; published 12 February 2009)

Probab. Theory Relat. Fields 120, 1–67 (2001)
Digital Object Identifier (DOI) 10.1007/s004400000115

G. Ben Arous · A. Dembo · A. Guionnet

Aging of spherical spin glasses

$$\mathcal{P}(\lambda_{\max} = t) \approx \begin{cases} \exp\left(-\beta N^2 \psi_-\left(\frac{t}{\sqrt{N}}\right) + \dots\right) & \text{for } t < \sqrt{2N} \text{ and } |t-\sqrt{2N}| \approx \mathcal{O}(N) \\ \frac{1}{a_\beta N^{-1/6}} F'_\beta\left(\frac{t-\sqrt{2N}}{a_\beta N^{-1/6}}\right) & \text{for } |t-\sqrt{2N}| \approx \mathcal{O}(N^{-1/6}) \\ \exp\left(-\beta N \psi_+\left(\frac{t}{\sqrt{N}}\right) + \dots\right) & \text{for } t > \sqrt{2N} \text{ and } |t-\sqrt{2N}| \approx \mathcal{O}(N) \end{cases}$$

$$\lim_{N\to\infty} \frac{1}{\beta N^2} \ln \mathcal{P}(\lambda_{\max} = z\sqrt{N}) = -\psi_-(z) \qquad \text{for } z < \sqrt{2}$$

$$\lim_{N\to\infty} \frac{1}{\beta N} \ln \mathcal{P}(\lambda_{\max} = z\sqrt{N}) = -\psi_+(z) \qquad \text{for } z > \sqrt{2}$$

Rare (extreme, atypical) fluctuations: large deviations

A simple example of large deviation tails

- Let $M \to$ no. of heads in N tosses of an unbiased coin
- Clearly $P(M,N) = \binom{N}{M} 2^{-N}$ $(M = 0, 1, \ldots, N) \to$ binomial distribution

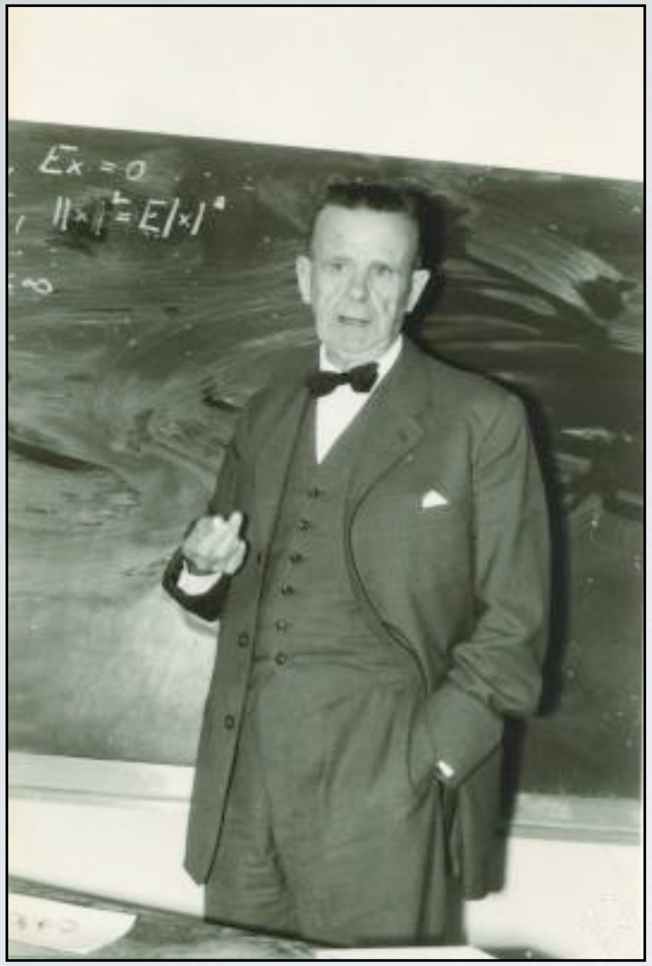
  with mean= $\langle M \rangle = \frac{N}{2}$ and variance=$\sigma^2 = \langle \left(M - \frac{N}{2}\right)^2 \rangle = \frac{N}{4}$
- typical fluctuations $M - \frac{N}{2} \sim O(\sqrt{N})$ are well described by the Gaussian form: $P(M,N) \sim \exp\left[-\frac{2}{N}\left(M - \frac{N}{2}\right)^2\right]$
- Atypical large fluctuations $M - \frac{N}{2} \sim O(N)$ are not described by Gaussian form
- Setting $M/N = x$ and using Stirling's formula $N! \sim N^{N+1/2} e^{-N}$ gives

$$P(M = Nx, N) \sim \exp\left[-N\Phi(x)\right] \qquad \text{where}$$

$$\boxed{\Phi(x) = x\log(x) + (1-x)\log(1-x) + \log 2} \to \text{large deviation function}$$

- $\Phi(x) \to$ symmetric with a minimum at $x = 1/2$ and for small arguments $|x - 1/2| << 1$, $\Phi(x) \approx 2(x - 1/2)^2$

  $\to$ recovers the Gaussian form near the peak



Harald Cramér
(Swedish mathematician)

Application to the insurance business

- Earning: constant rate per month (the monthly premium)
- The claims X_i come randomly

For the company to be successful over a certain period of time (preferably many months), the total earning should exceed the total claim.
Thus to estimate the premium you have to ask the following question :
"What should we choose as the monthly premium q such that over N months the sum of the claims is less than Nq?"
Cramér gave a solution to this question for i.i.d. random variables...

A Trivial Problem

DIAGONAL MATRIX

$$\begin{bmatrix} x_{11} & & & & \\ & x_{22} & & 0 & \\ & & x_{33} & & \\ & 0 & & \ddots & \\ & & & & x_{NN} \end{bmatrix}$$

$$\Pr[x_{ii} = x] = (2\pi)^{-1/2} \exp[-x^2/2]$$

$\downarrow$

GAUSSIAN

N Eigenvalues: $\lambda_i = x_{ii} \longrightarrow$ Independent

- $P_N = \text{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = 2^{-N} = \exp[-(\ln 2)\, N]$

A Nontrivial Problem

REAL SYMMETRIC MATRIX (NxN)

$$X = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & & \vdots \\ x_{N1} & \cdots & \cdots & x_{NN} \end{vmatrix}$$

GAUSSIAN → $\Pr[X] \propto \exp[-\frac{1}{2}\mathrm{Tr}(X^2)]$

N eigenvalues : $\lambda_1, \lambda_2, \ldots, \lambda_N$ → strongly correlated

- $P_N = \mathrm{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = \mathrm{Prob}[\lambda_{\max} \leq 0] = ?$

[R.M. May, Nature, 238, 413 (1972)—Ecosystems]

[Cavagna et. al. 2000, Fyodorov 2004, —— Glassy systems]

[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006— String theory].....

$N = 5$

$$\begin{pmatrix} 0.5377 & 0.2631 & -1.8044 & 0.3286 & 0.4951 \\ 0.2631 & -0.4336 & 1.6888 & 1.7271 & 0.7810 \\ -1.8044 & 1.6888 & 0.7254 & 0.7133 & 0.7160 \\ 0.3286 & 1.7271 & 0.7133 & 1.4090 & 1.5237 \\ 0.4951 & 0.7810 & 0.7160 & 1.5237 & 0.4889 \end{pmatrix}$$

$$\vec{\lambda} = \begin{bmatrix} -2.4341 & -0.8386 & -0.5203 & 2.2594 & 4.2610 \end{bmatrix}$$

All the eigenvalues negative??
Can it ever happen?

Results for P_N:

- $P_N = \mathrm{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = ?$
- $N = 1$: $P_1 = 1/2 = 0.5$ (trivially)
- $N = 2$: $P_2 = \frac{2-\sqrt{2}}{4} = 0.146447..$
- $N = 3$: $P_3 = \frac{\pi - 2\sqrt{2}}{4\pi} = 0.0249209..$

(Beltrani 2007, Dedieu & Malajovich, 2007)

Question: How does P_N decay for large N, i.e., $P_N \to ?$ as $N \to \infty$

- Based on numerics, Aazami & Easther (2006) predicted for large N:

$$\boxed{P_N \sim \exp[-\theta N^2]} \text{ with } \theta_{\mathrm{num}} \approx 0.27$$

$\to$ very small probability $\to$ RARE EVENT

Logarithmic equivalence

<u>Applications?</u>

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Cosmology from random multifield potentials

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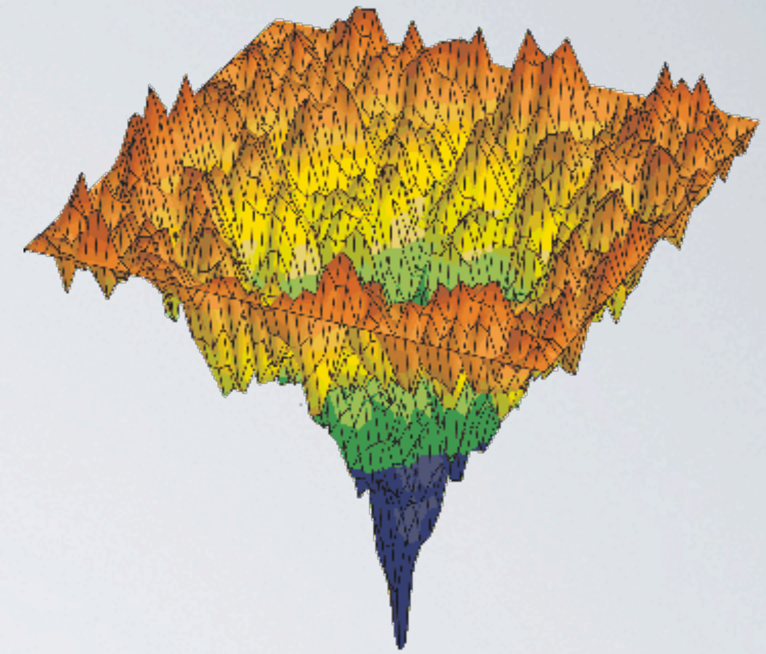
Received 23 January 2006



Despite the approximation used to obtain equation (8), we have confirmed that the likelihood that all the eigenvalues of an $N \times N$ symmetric matrix have the same sign scales as e^{-cN^2}. The measured constant differs slightly from -0.25, although given the simplicity of our approximation the agreement is perhaps surprisingly good.

- Based on numerics, Aazami & Easther (2006) predicted for large N:

$$\boxed{P_N \sim \exp[-\theta N^2]} \text{ with } \theta_{\rm num} \approx 0.27$$

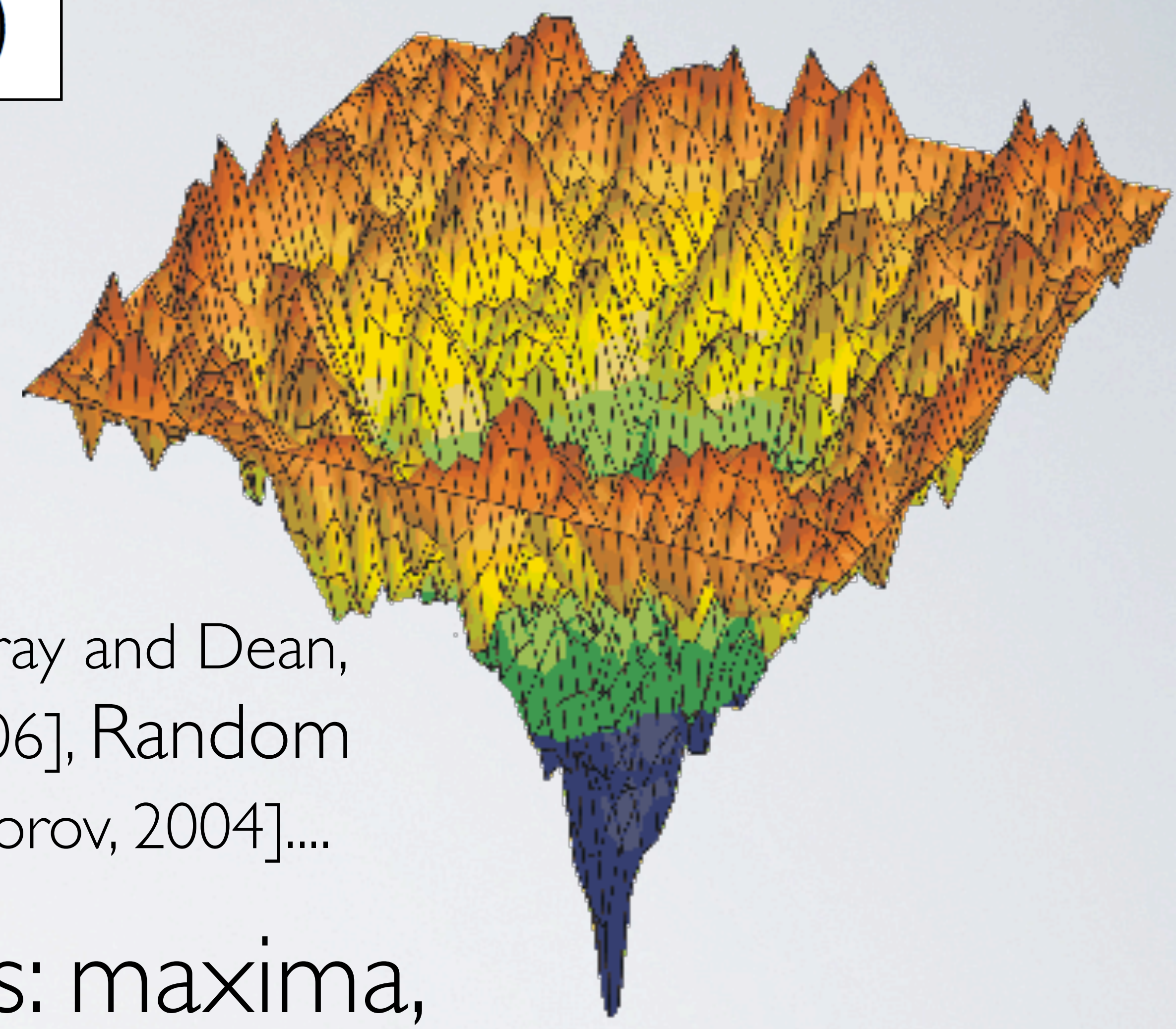
$\rightarrow$ very small probability $\rightarrow$ RARE EVENT

- Exact result: $\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..}$ (Dean and S.M., 2006)

More generally, for $\beta = 1$ (GOE), $\beta = 2$ (GUE) and $\beta = 4$ (GSE)

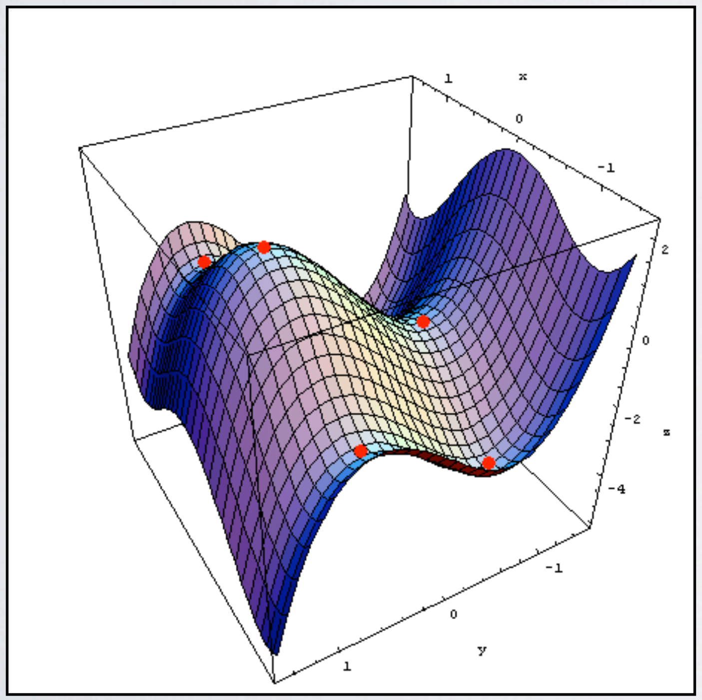
$$\boxed{P_N \sim \exp[-\beta\theta N^2]} \text{ for large } N$$

A particle moving in a N-dim. landscape $V(y_1, \ldots, y_N)$



$$\frac{dy_i}{dt} = -\nabla_{y_i} V$$

- Spin and structural glasses, Gaussian fields [Bray and Dean, 2006], String landscapes [Aazami and Easther, 2006], Random Energy Landscapes and Glass Transition [Fyodorov, 2004]....



Stationary points: maxima, minima and saddles

$$H_{i,j} = \left[\frac{\partial^2 V}{\partial y_i \partial y_j} \right]$$

Hessian matrix

Eigenvalues of Hessian matrix determine the nature of the stationary point

Eigenvalues of the Hessian Matrix

Examples:

- $N = 1$-dimensional surface: Hessian matrix $H = \frac{\partial^2 V}{\partial y^2}$

  If $\frac{\partial^2 V}{\partial y^2} < 0 \rightarrow$ Local Maximum; if $\frac{\partial^2 V}{\partial y^2} > 0 \rightarrow$ Local Minimum

- $N = 2$-dimensional surface: Hessian matrix $H \equiv \begin{pmatrix} \frac{\partial^2 V}{\partial y_1^2} & \frac{\partial^2 V}{\partial y_1 \partial y_2} \\ \frac{\partial^2 V}{\partial y_2 \partial y_1} & \frac{\partial^2 V}{\partial y_2^2} \end{pmatrix}$

  Two real eigenvalues: (λ_1, λ_2)

  If $\lambda_1 < 0$ and $\lambda_2 < 0 \rightarrow$ Local Maximum

  If $\lambda_1 > 0$ and $\lambda_2 > 0 \rightarrow$ Local Minimum

  $$\left.\begin{matrix} \lambda_1 < 0, & \lambda_2 > 0 \\ \lambda_1 > 0, & \lambda_2 < 0 \end{matrix}\right\} \quad \rightarrow \text{Saddle}$$

Random Hessian Model

Draw the elements of the Hessian matrix independently at random

$$H_{i,j} = \left[\frac{\partial^2 V}{\partial y_i \partial y_j} \right]$$

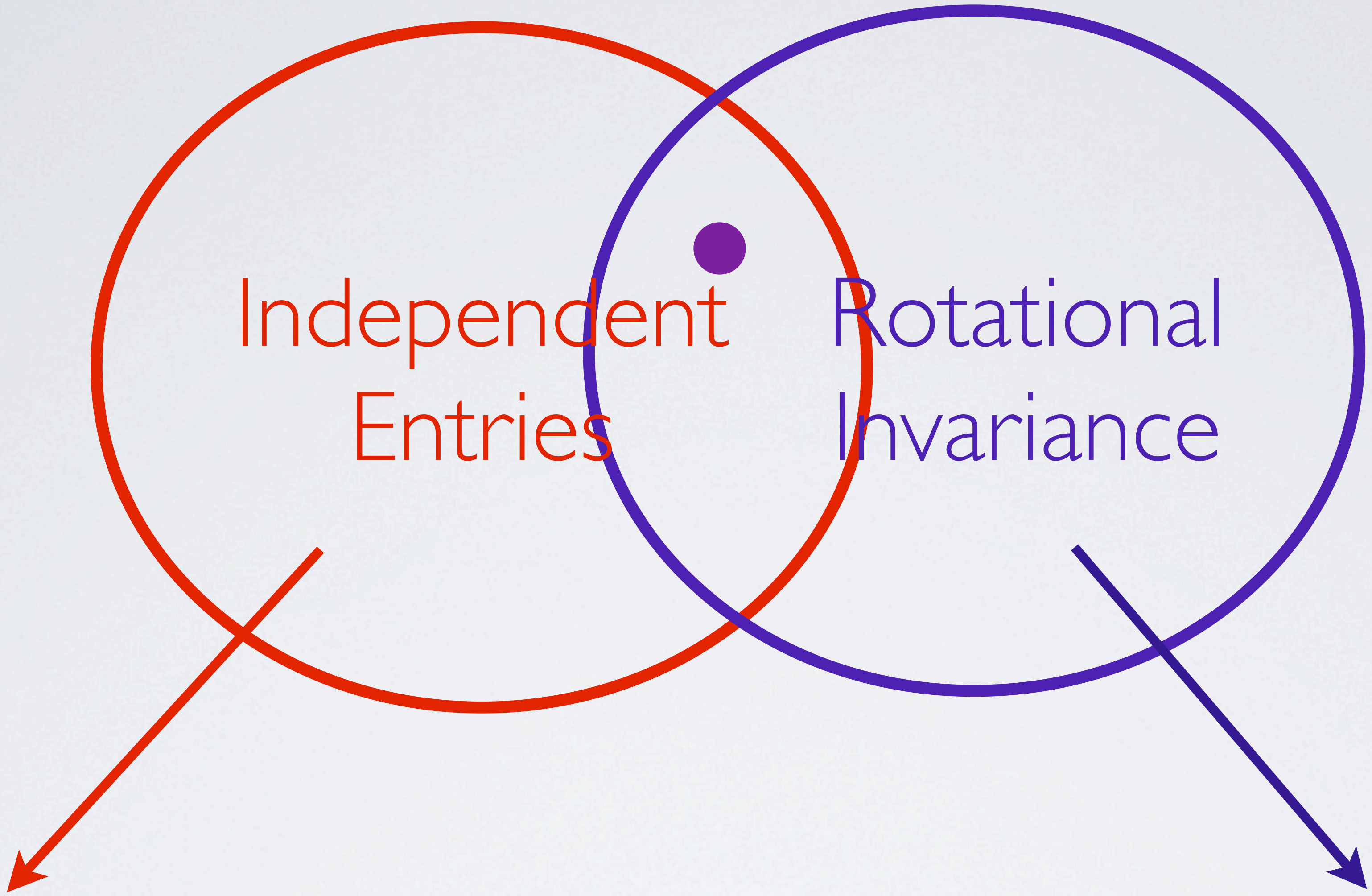
It belongs to the **GOE** of random matrices

The probability that **all** the eigenvalues are positive (or negative) provides information about the number and nature of extremal points

Most of the stationary points are **saddles**!

Techniques

Techniques



- Edwards-Jones formula
- Moments method

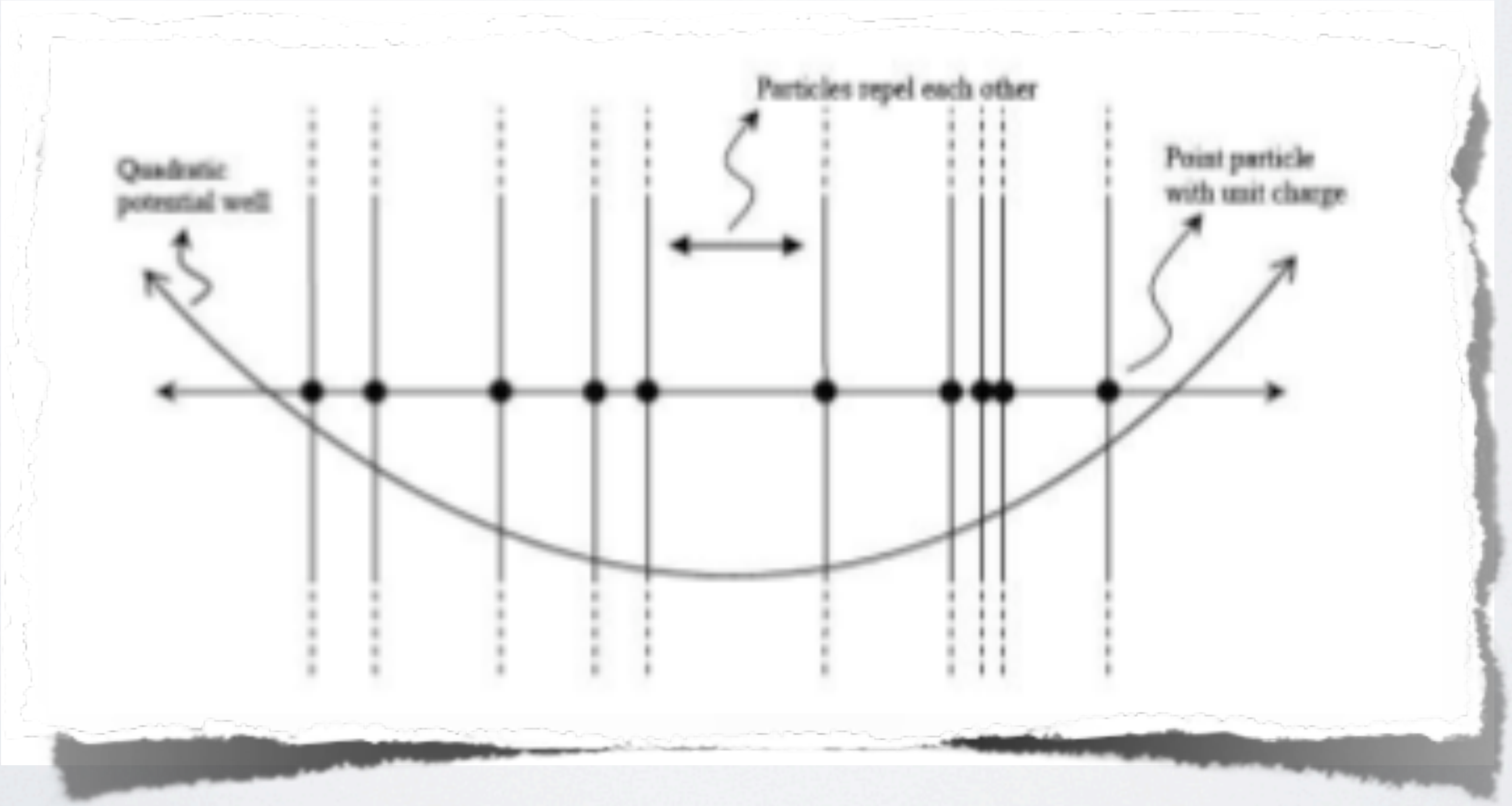
- Andreief formula
- Orthogonal Polynomials
- Coulomb gas

Coulomb gas

$$P_\beta(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N} e^{-\frac{\beta}{2}\sum_{i=1}^N \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta = \frac{1}{Z_N} e^{-\beta \mathcal{H}(\vec{\lambda})}$$

$$\mathcal{H}(\vec{\lambda}) = \frac{1}{2}\sum_{i=1}^N \lambda_i^2 - \frac{1}{2}\sum_{j\neq k} \log|\lambda_j - \lambda_k|$$

Canonical weight of an auxiliary thermodynamical system



Dyson?

STATISTICAL PROPERTIES OF REAL SYMMETRIC MATRICES WITH MANY DIMENSIONS

E. P. WIGNER, *Princeton University*

[*Can. Math. Congr. Proc.*, Toronto 1957]

If the density of the roots at λ is $\sigma(\lambda)$, the logarithm of the probability P is given by

$$(6) \quad \ln P(\lambda_1, \lambda_2, \ldots, \lambda_n) = \text{const} - \sum_i \tfrac{1}{4}\lambda_i^2 + \sum_{i<k} \ln|\lambda_i - \lambda_k|.$$

It can be approximated by the following functional of σ

$$(6a) \quad [\sigma] = \text{const} - \tfrac{1}{4}\int d\lambda\, \lambda^2\sigma(\lambda) + \tfrac{1}{2}\int d\lambda \int d\mu\, \sigma(\lambda)\sigma(\mu) \ln|\lambda - \mu|.$$

All integrations have to be extended from $-\infty$ to ∞ and σ is so normalized that

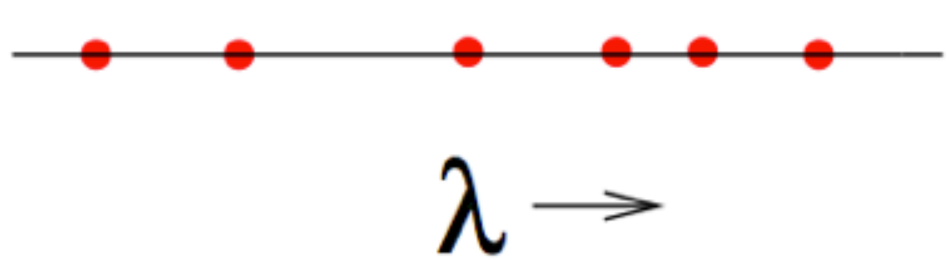
$$(7) \quad \int \sigma(\lambda) d\lambda = n.$$

Distribution of top eigenvalue: Coulomb gas method

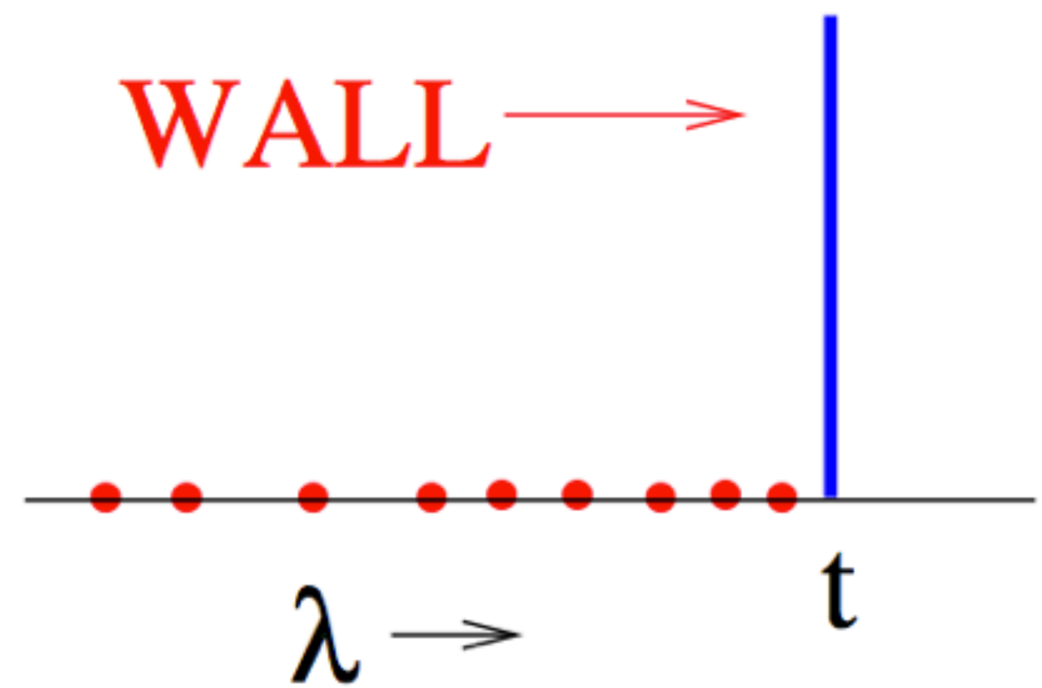
$$\mathrm{Prob}[\lambda_{\max} \le t,\, N] = \mathrm{Prob}[\lambda_1 \le t, \lambda_2 \le t, \ldots, \lambda_N \le t] = \frac{Z_N(t)}{Z_N(\infty)}$$

$$\boxed{Z_N(t) = \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]}$$

denominator



numerator



Work on scale $\lambda \sim \sqrt{N}$ with large N

- Scaled variables: $x_i = \lambda_i/\sqrt{N}$; maximum x_i: $w = t/\sqrt{N}$
- $$Z_N(w) \propto \int_{-\infty}^{w} \prod_i dx_i \exp\left[-\beta N^2 E\left(\{x_i\}\right)\right]$$

$$E\left(\{x_i\}\right) = \frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{j\neq k}\log|x_j - x_k|$$

- Introduce counting function (scaled density): $f(x) = \frac{1}{N}\sum_i \delta(x - x_i)$
- discrete sum $\rightarrow$ continuous integral:

$$E[f(x)] = \int_{-\infty}^{w} x^2 f(x)\,dx - \int_{\infty}^{w}\int_{-\infty}^{w} \ln|x - x'|\, f(x)\, f(x')\, dx\, dx'$$

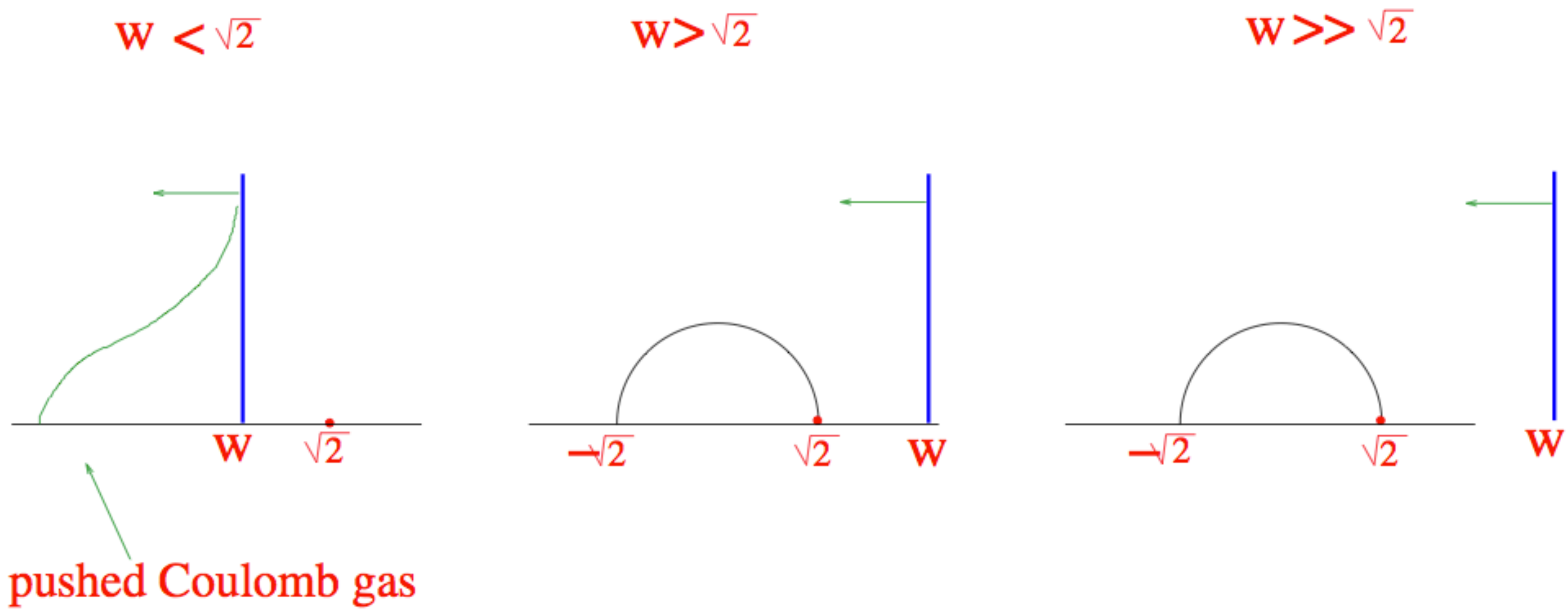
$$Z_N(w) \propto \int \mathcal{D}f(x)\,\exp\left[-\beta N^2\left\{E[f(x)] + C\left(\int f(x)dx - 1\right)\right\} + O(N)\right]$$

- for large N, minimize the action $S[f(x)] = E[f(x)] + C\left[\int f(x)dx - 1\right]$
  Saddle Point Method: $\frac{\delta S}{\delta f} = 0 \rightarrow f_w(x) \rightarrow$

$$Z_N(w) \sim \exp\left[-\beta N^2 S\left[f_w(x)\right]\right]$$

As we bring the wall from ∞

charge density $f_w(x)$ vs. x for different w



$w = \sqrt{2}$ ⟶ CRITICAL POINT

Saddle Point Solution

- saddle point $\frac{\delta S}{\delta f} = 0 \rightarrow$ singular integral Eq. for $f_w(x)$

- $\boxed{x = \mathcal{P}\int_{-\infty}^{w} \frac{f_w(y)\,dy}{x-y}}$ for $x \in [-\infty, w] \rightarrow$ Semi-Hilbert transform

$\rightarrow$ Inverse electrostatic problem $\rightarrow$ Given the potential x find the charge density $f_w(x)$ (though not quite!)

- General method for solving such singular integral equations $\rightarrow$ Tricomi (1957)

Tricomi Solution

Assuming finite support of $f(x)$ over $[a, b]$

- $$U(x) = \mathcal{P}\int_a^b \frac{f(y)\,dy}{x-y}$$ for $x \in [a, b]$

- General solution (Tricomi, '57):

Equazioui integrali singolari del tipo di Carleman.

Francesco G. Tricomi (a Torino).

A Mauro Picone nel suo 70mo compleanno.

$$f(x) = -\frac{1}{\pi^2\sqrt{(b-x)(x-a)}}\left[\mathcal{P}\int_a^b \frac{\sqrt{(b-x')(x'-a)}}{x-x'}\,U(x')\,dx' + B\right]$$

for $x \in [a, b]$

where $B \rightarrow$ arbitrary constant

- In our problem, $U(x) = x$ and $b = w$ (wall position) and we assume $a = -L_1(w)$

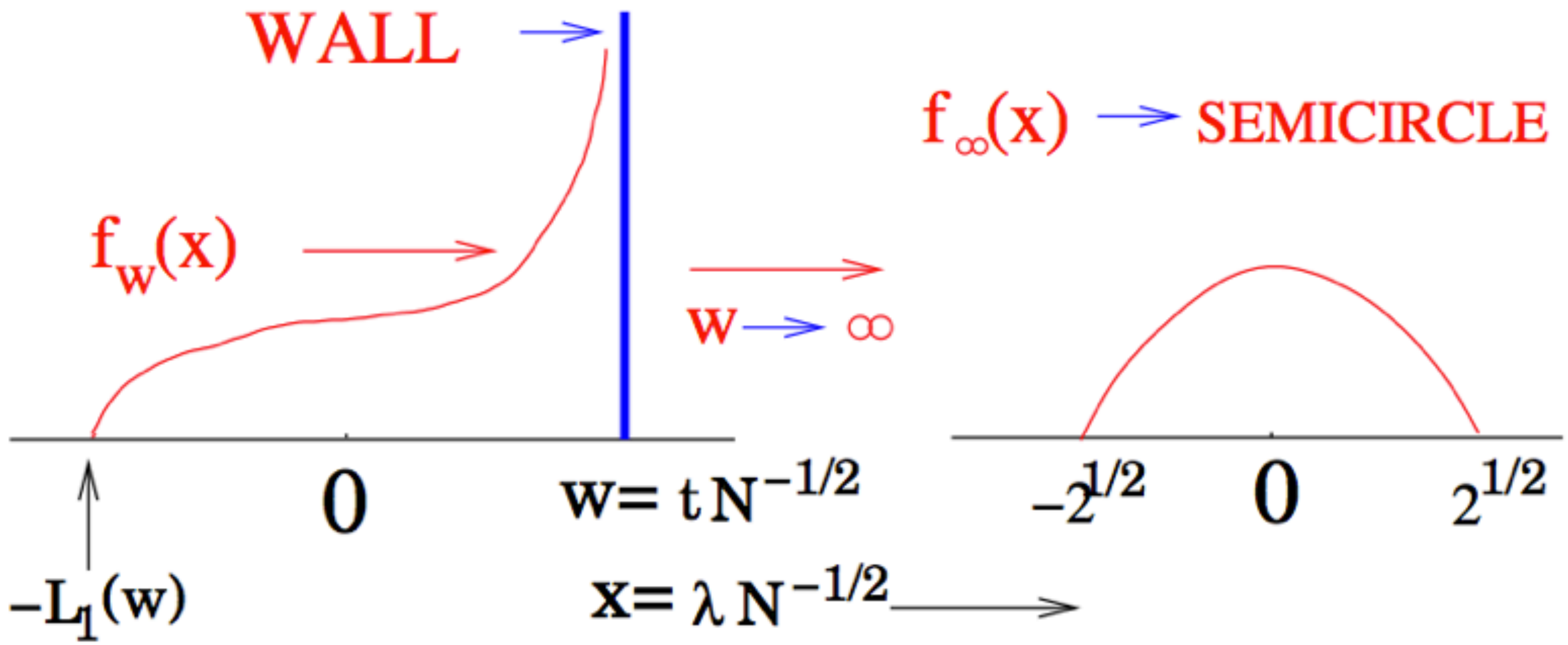
Exact Saddle Point Solution

- Exact solution *[D. Dean and S.N. Majumdar, 2008]*

$$f_w(x) = \frac{\sqrt{x + L_1(w)}}{2\pi\sqrt{w - x}}\,[w + L_1(w) - 2x]$$

where $-L_1(w) \leq x \leq w$ and $L_1(w) = [2\sqrt{w^2+6} - w]/3$

- When $w \to \infty$, $L_1(w) \to \sqrt{2}$ and $f_\infty(x) = \sqrt{2 - x^2}/\pi \to$ semicircle



Left large deviation function

- $$\mathrm{Prob}[\lambda_{\max} \le t, N] = \frac{Z_N(t)}{Z_N(\infty)}$$
$$\sim \exp\left[-\beta N^2 \left\{ S[f_{w=t/\sqrt{N}}(x)] - S[f_\infty(x)] \right\}\right]$$
$$\sim \exp\left[-\beta N^2 \, \Phi_-\left(\frac{t}{\sqrt{N}}\right)\right]$$

- where $\Phi_-(w)$ (for $w < \sqrt{2}$) is the left large deviation function physically $\Phi_-(w) =$ energy cost in pushing the Coulomb gas

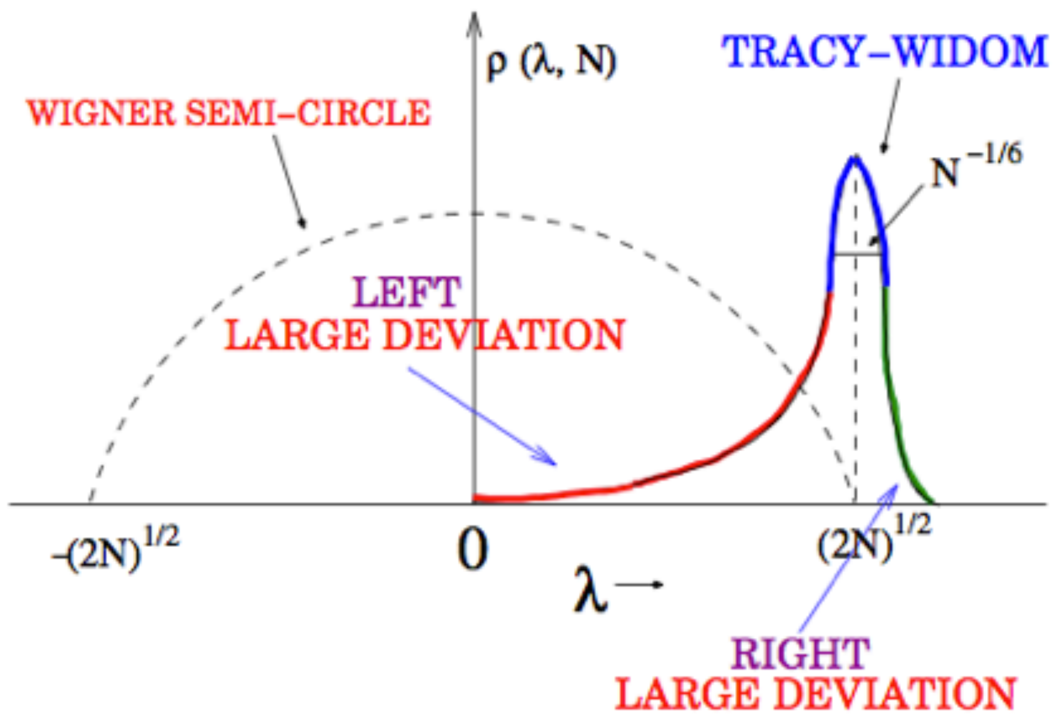
$$\Phi_-(w) = \frac{1}{108}\left[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} + 27\left(\ln(18) - 2\ln(w + \sqrt{6+w^2})\right)\right] \quad \text{where} \quad w < \sqrt{2}$$

- In particular, setting $w = 0$, $P_N \sim \exp[-\beta\theta N^2]$

$$\boxed{\theta = \Phi_-(0) = \frac{1}{4}\ln(3) = 0.274653\ldots}$$

[D. Dean and S.N. Majumdar, 2008]

Matching the left tail of Tracy-Widom distribution:

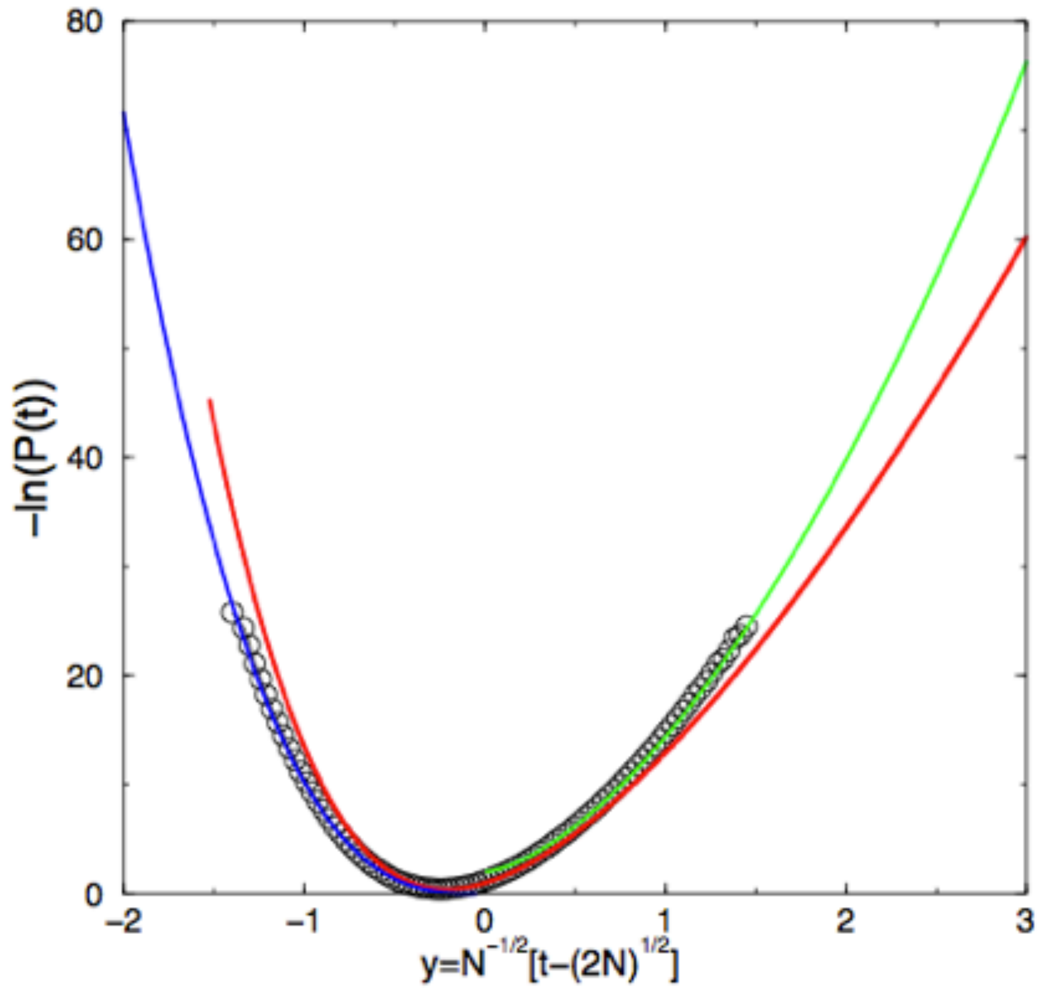


- $\mathrm{Prob}[\lambda_{\max} \leq t, N] \sim \exp\left[-\beta N^2\, \Phi_-\left(\frac{t}{\sqrt{N}}\right)\right]$; $\quad w = \frac{t}{\sqrt{N}}$
- When $w \to \sqrt{2}$ from below, $\to$ left tail of Tracy-Widom
- As $w \to \sqrt{2}$ from below, $\Phi_-(w) \to \frac{(\sqrt{2}-w)^3}{6\sqrt{2}} \Rightarrow$

$$\mathrm{Prob}[\lambda_{\max} \leq t, N] \approx \exp\left[-\frac{\beta}{24}\left|\sqrt{2}\, N^{1/6}\,(t-\sqrt{2N})\right|^3\right]$$

- recovers the correct left tail of TW: $F_\beta(x) \sim \exp[-\frac{\beta}{24}\,|x|^3]$ as $x \to -\infty$

Comparison with Simulations:



[S.N. Majumdar and M. Vergassola, 2009]

$N \times N$ real Gaussian matrix ($\beta = 1$): $N = 10$

circles $\longrightarrow$ simulation points

red line $\longrightarrow$ Tracy-Widom

blue line $\longrightarrow$ left large deviation function ($\times N^2$)

green line $\longrightarrow$ right large deviation function ($\times N$).

SUMMARY

- Extreme Value Theory for i.i.d. and correlated random variables
- Tracy-Widom distribution: **ubiquitous**!
- Connection with field theories and models of statistical mechanics (non-intersecting BM, LIS...)
- Experimental verification (KPZ, coupled lasers...)
- Rare events and large deviations: the case of the largest eigenvalue with Coulomb gas technique