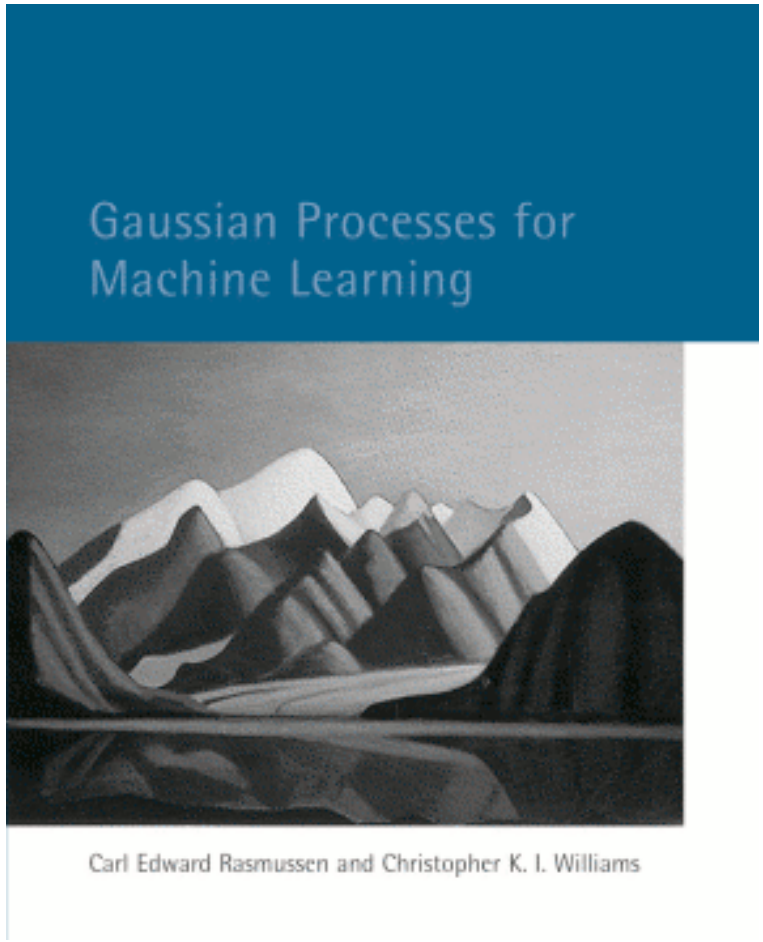
Gaussian Processes

Edwin V. Bonilla

Machine Learning Summer School

October 1st, 2010

The Book

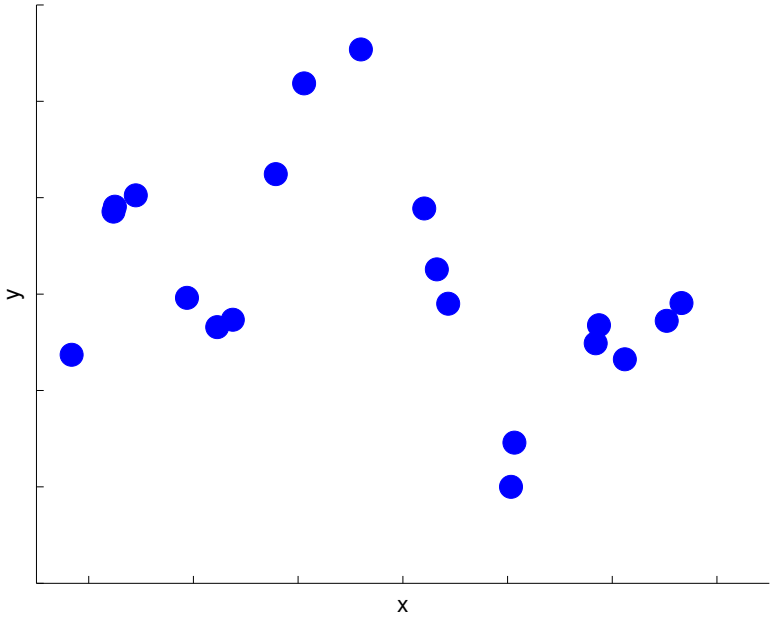


Carl Edward Rasmussen and Christopher K. I. Williams

All chapters available online along with software and datasets:
http://www.gaussianprocess.org/gpml

The Prediction Problem

Learn mapping $\mathbf{x} \rightarrow \mathrm{f}(\mathbf{x})$ from observations $\{(\mathbf{x_i}, \mathbf{y_i})\}_{\mathbf{i}=1}^{\mathbf{N}}$.



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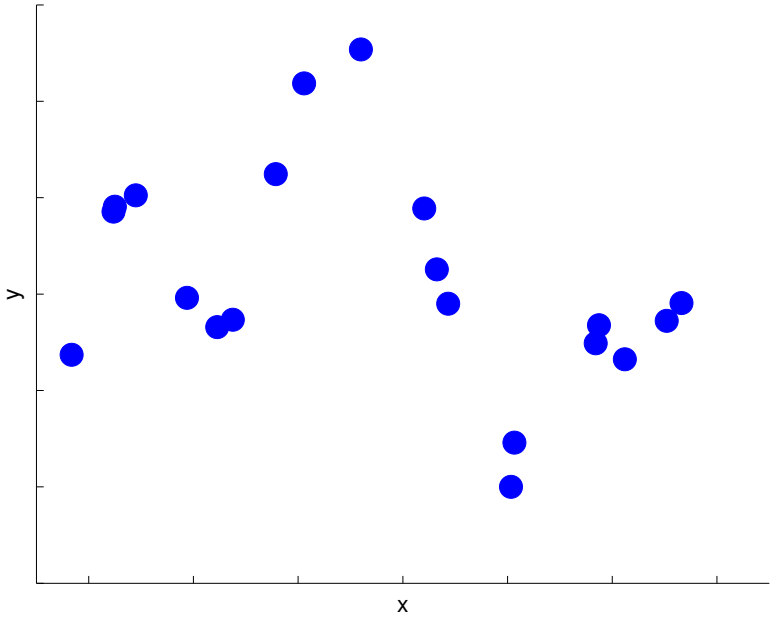

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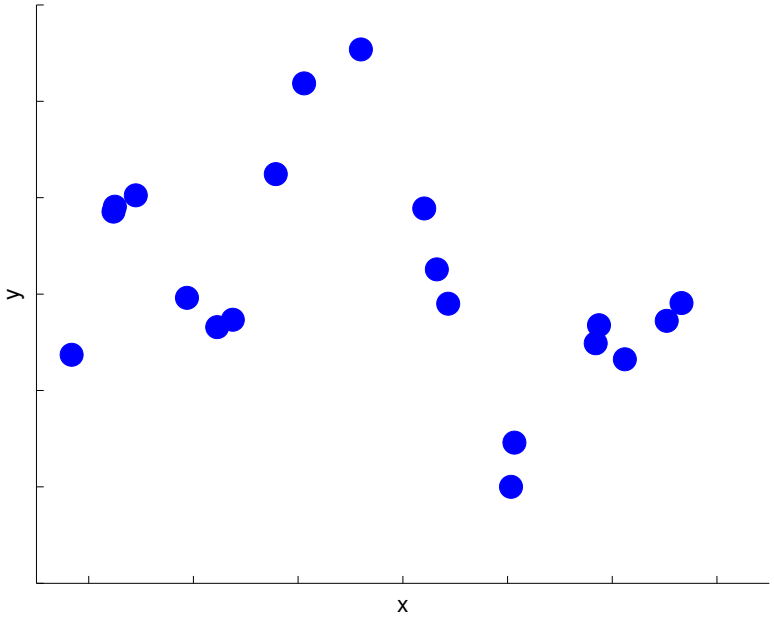

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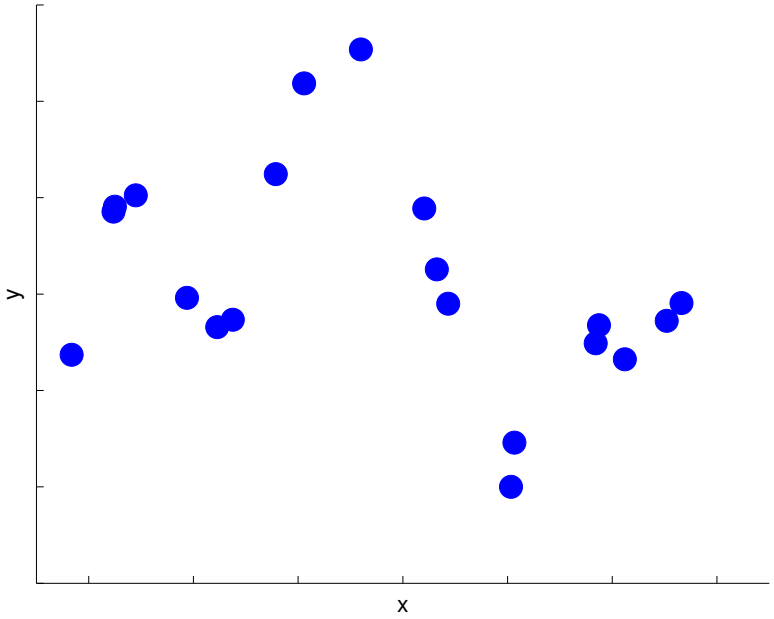

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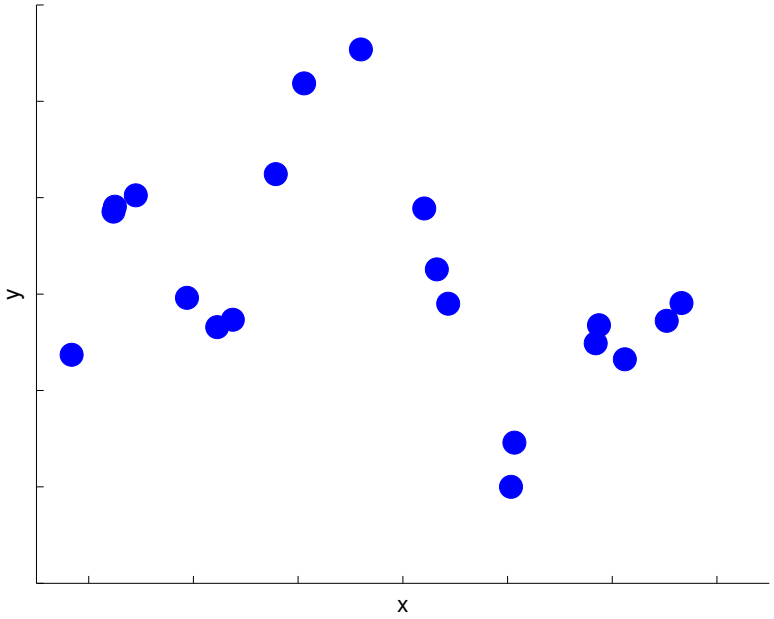

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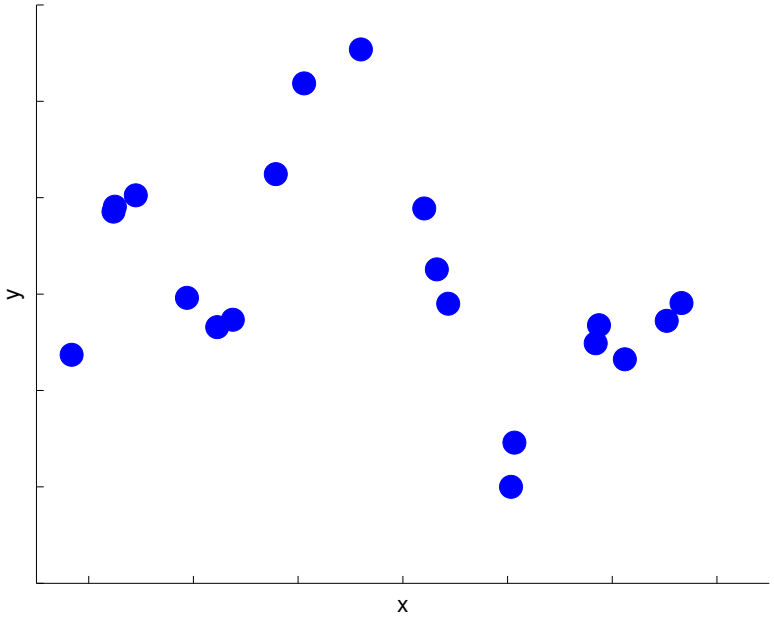

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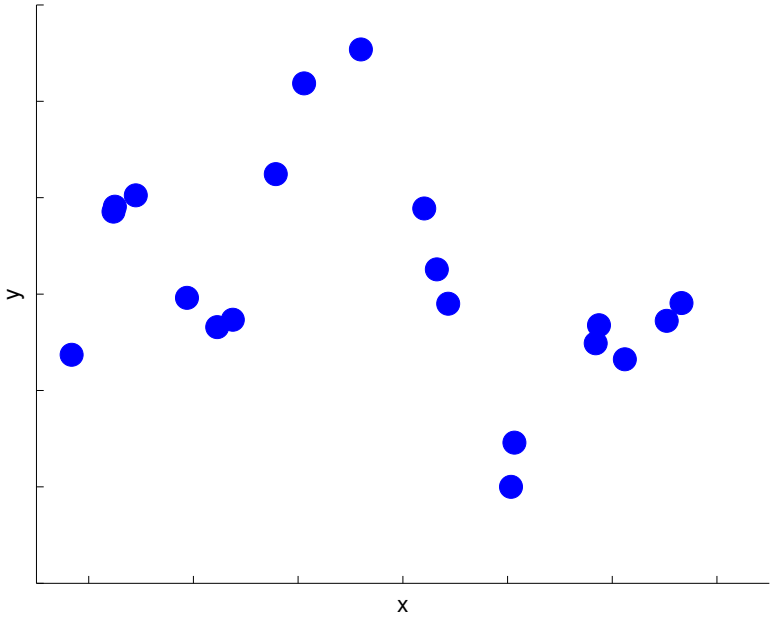

- What about Neural nets?

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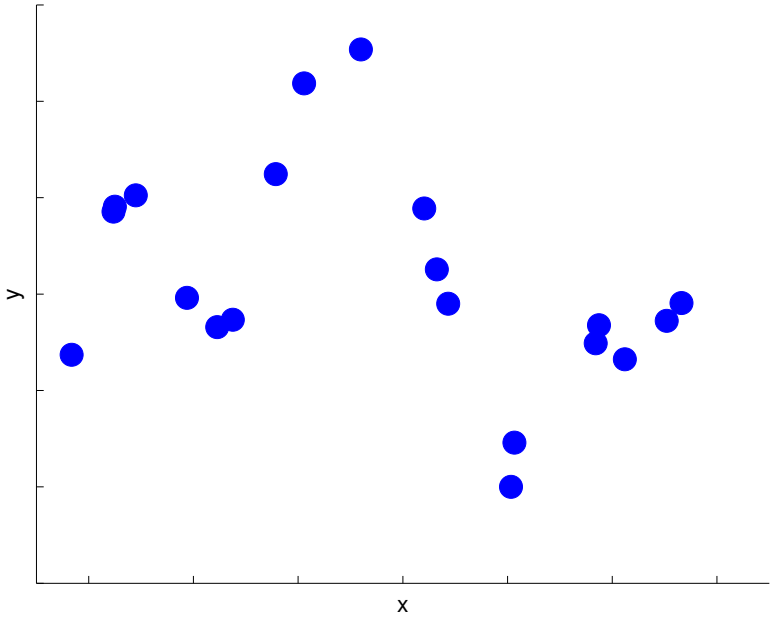

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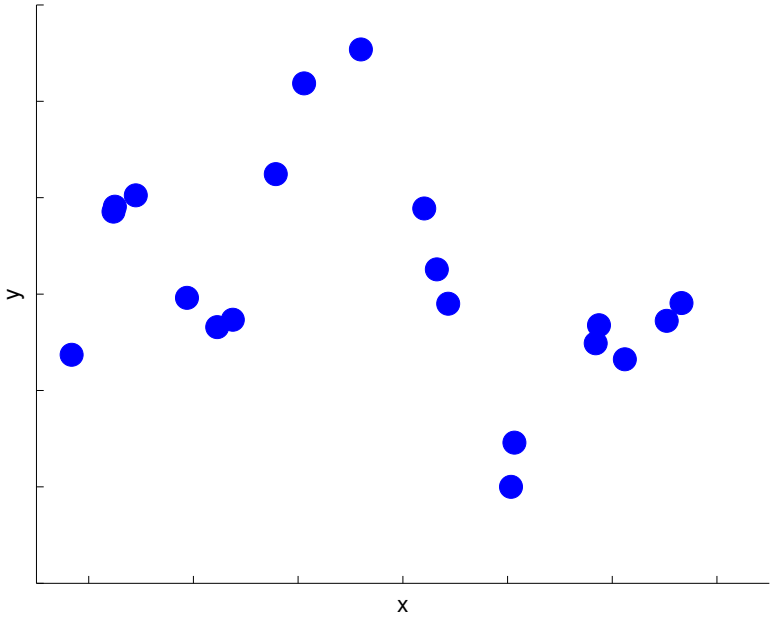

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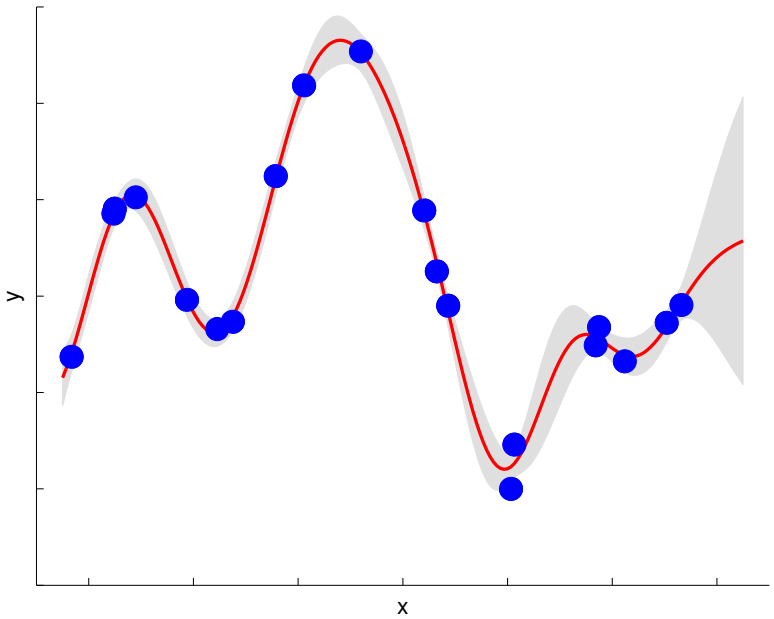

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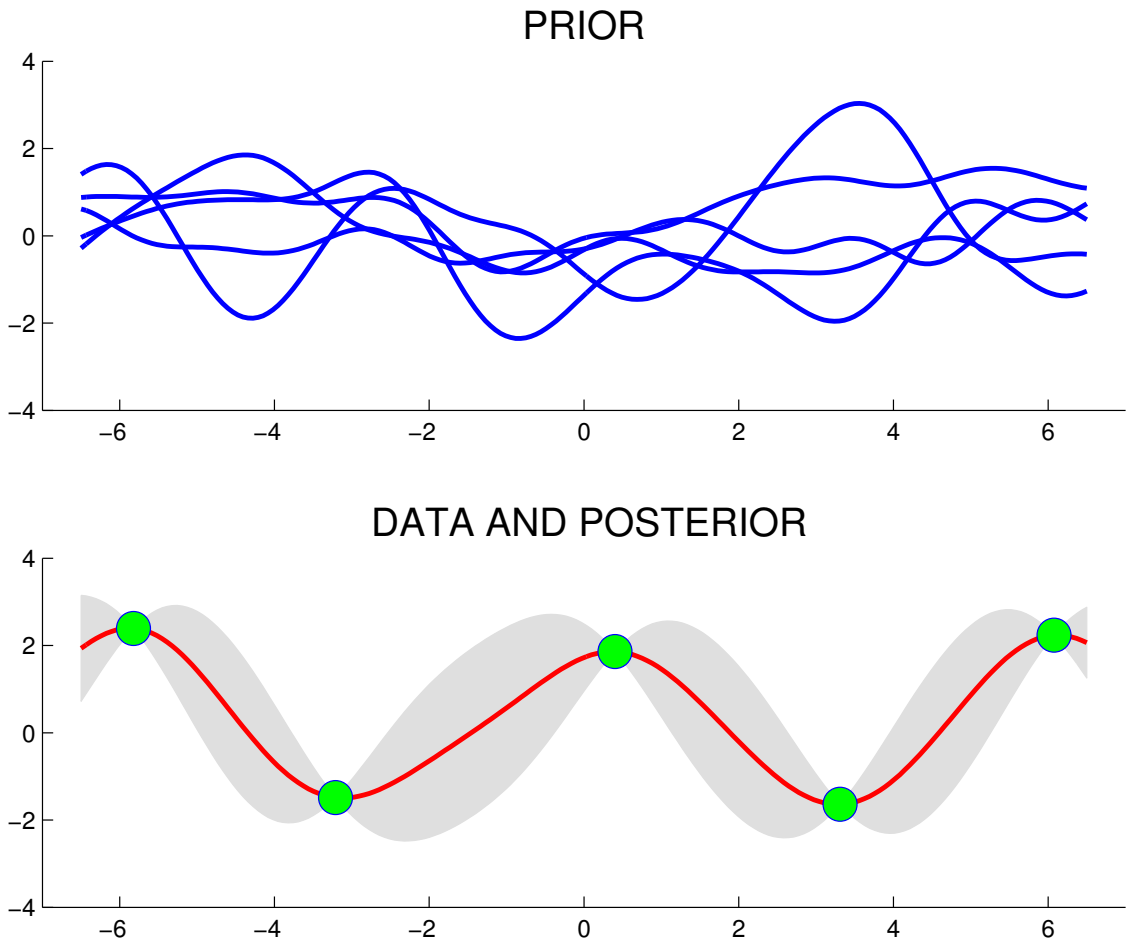
We can address these issues in a principled way with Gaussian Processes

Demo



- Smooth functions
- Closeness in input space $\rightarrow$ closeness in output space

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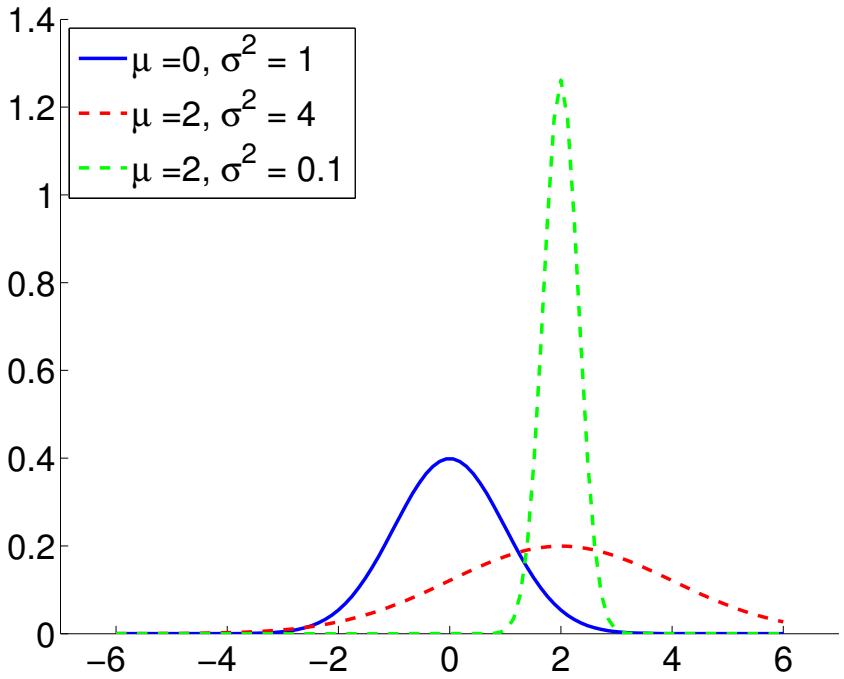
- Many standard regression models are special cases of GPs
- GP models also applicable to non-regression settings

Outline

The Gaussian Distribution

1D Example



$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

The Gaussian Distribution

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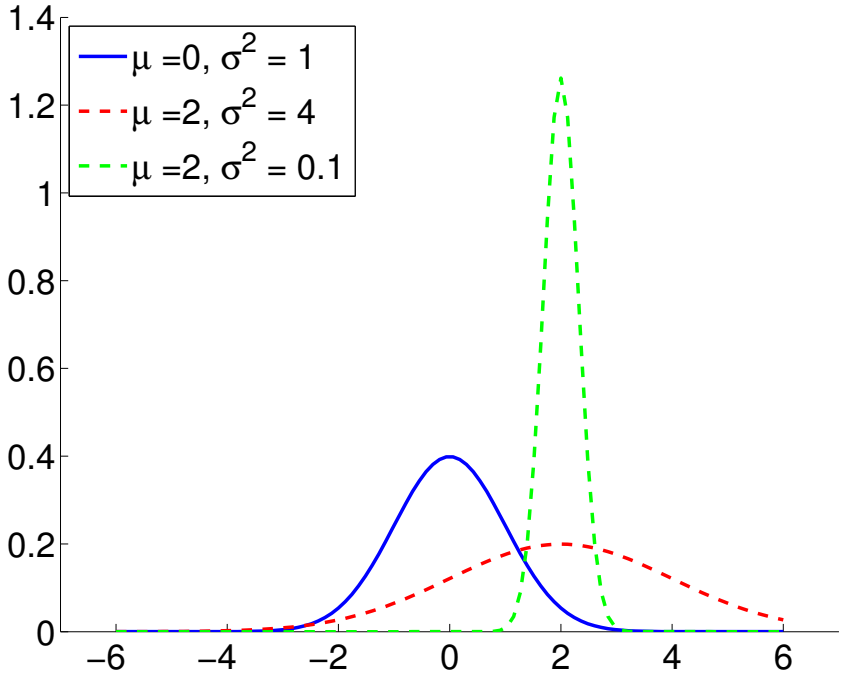

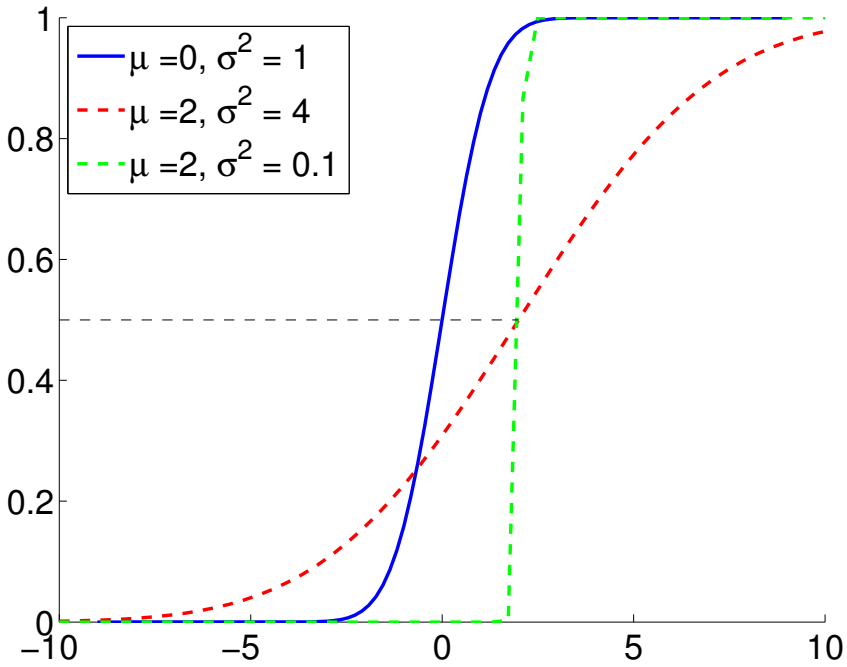

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The Gaussian Distribution

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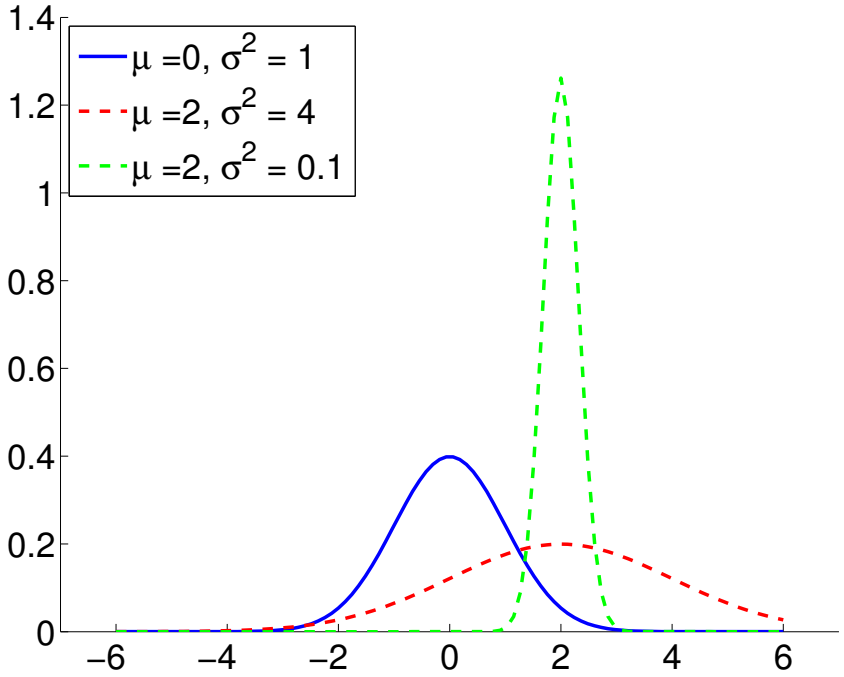

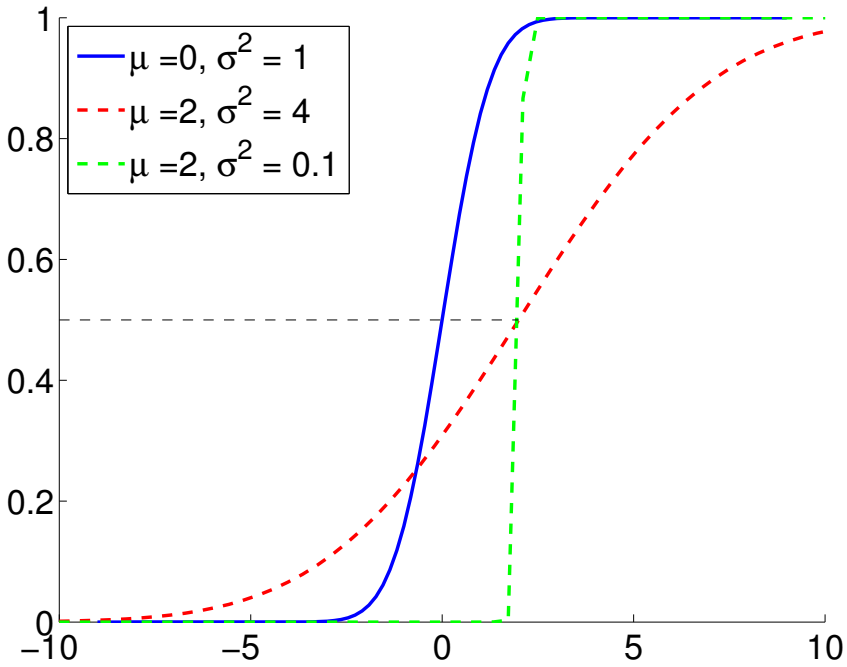

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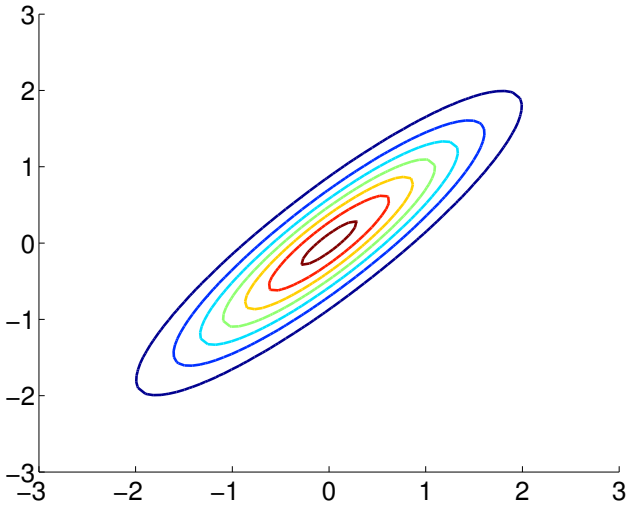
In general: $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{|2\pi\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$

The Gaussian Distribution

2D Example

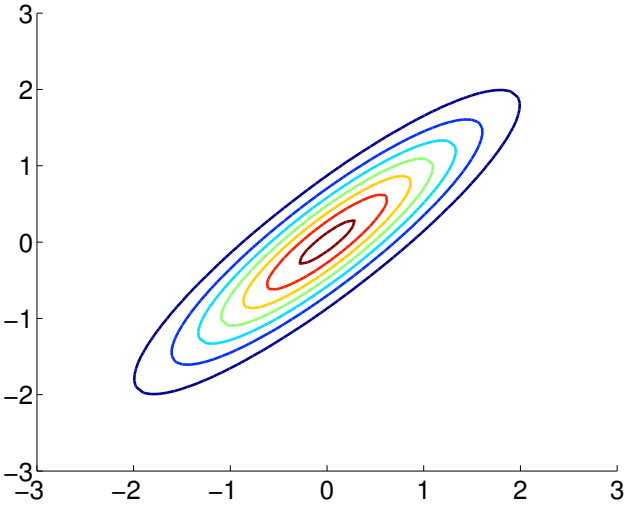


$$p(x_1, x_2) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Joint

The Gaussian Distribution

2D Example



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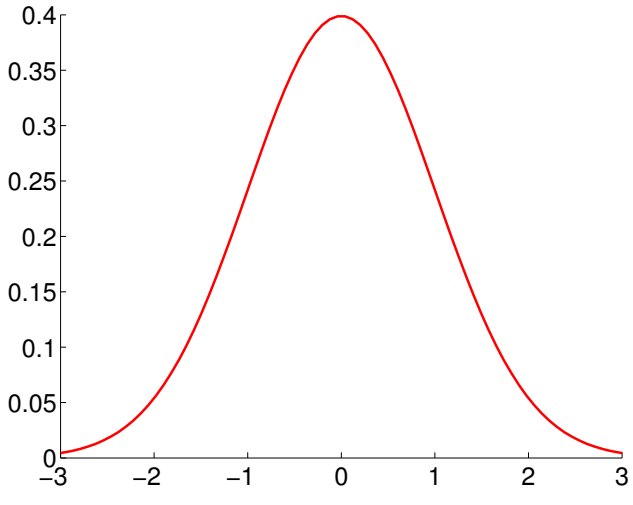
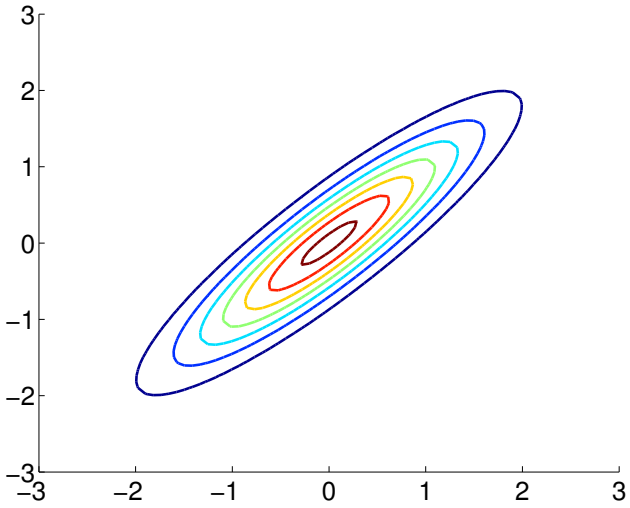


$$p(x1)$$

Marginal

The Gaussian Distribution

2D Example



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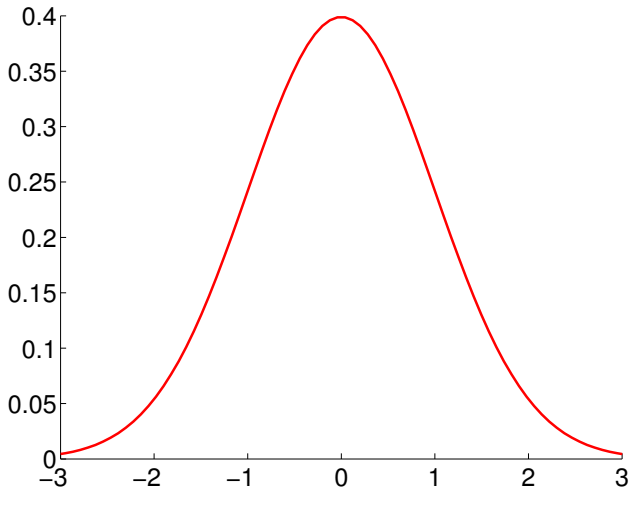

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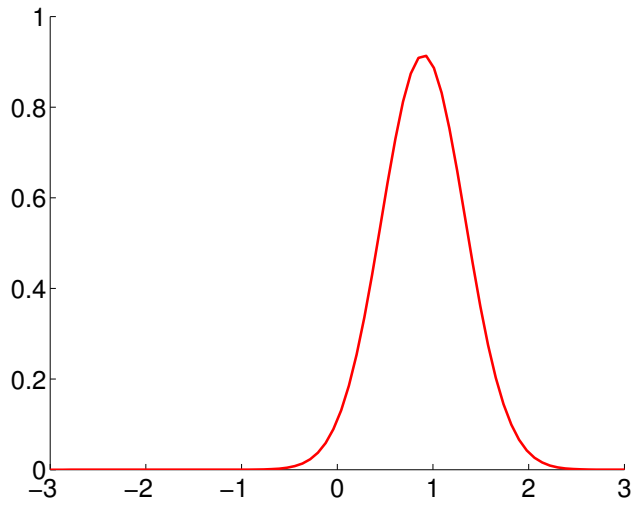

$$p(x_1|x_2)$$

Conditional

The Gaussian Distribution

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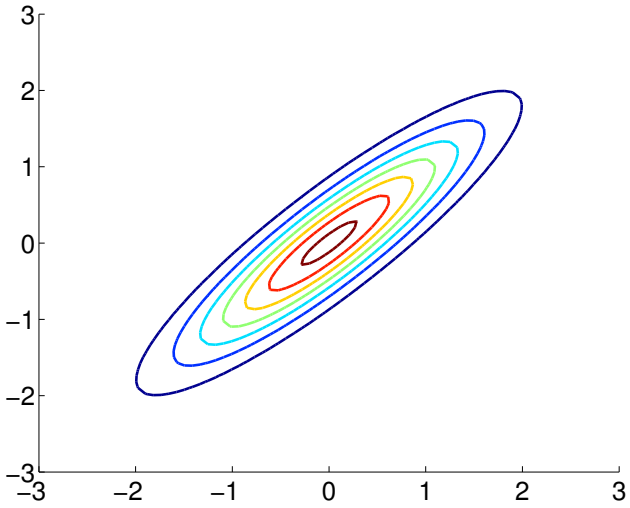

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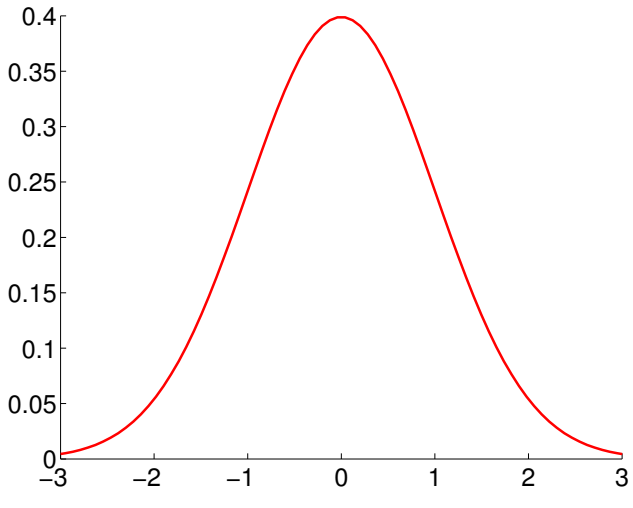

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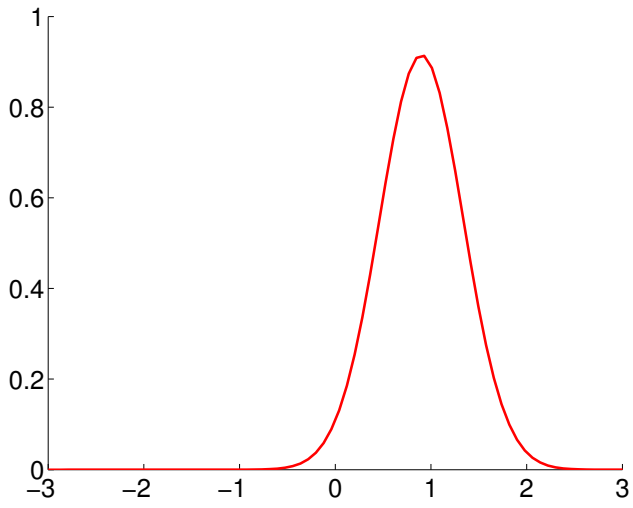

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The marginal and the conditional distributions are also Gaussians:

The Gaussian Distribution

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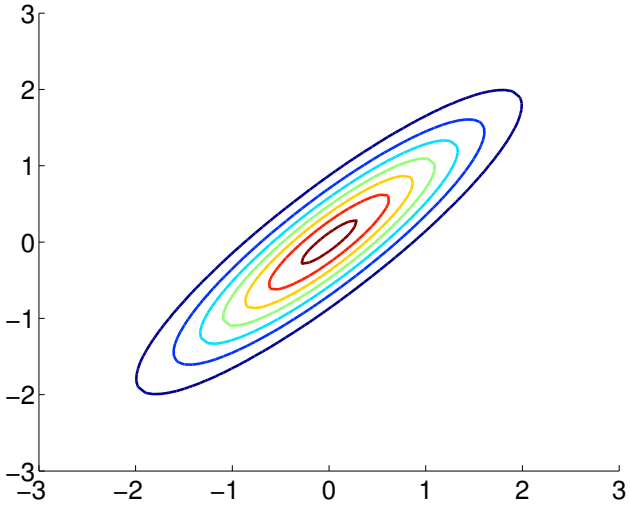

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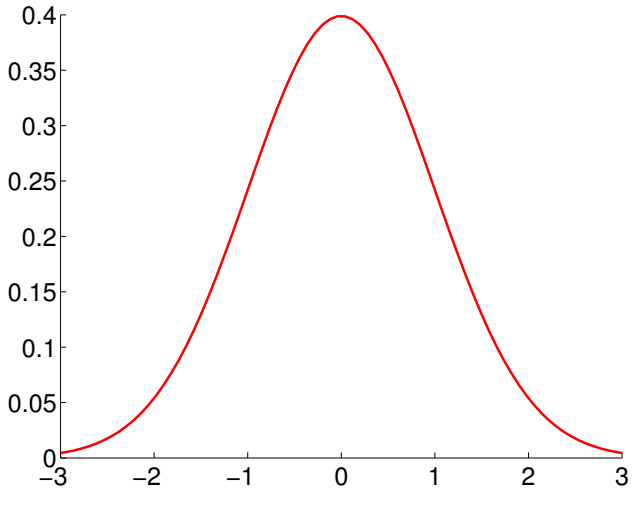

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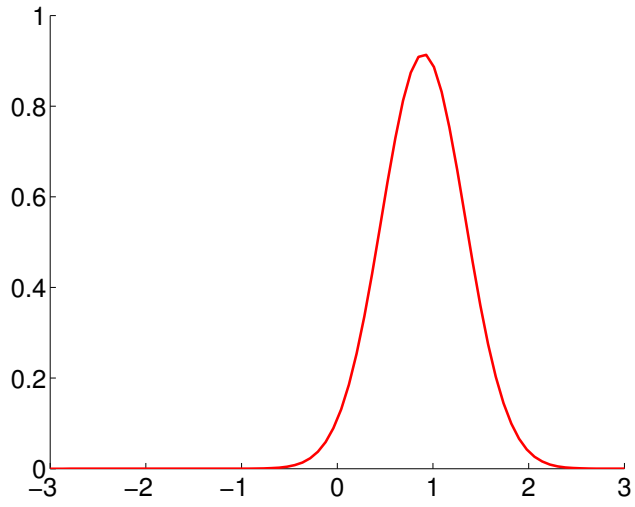

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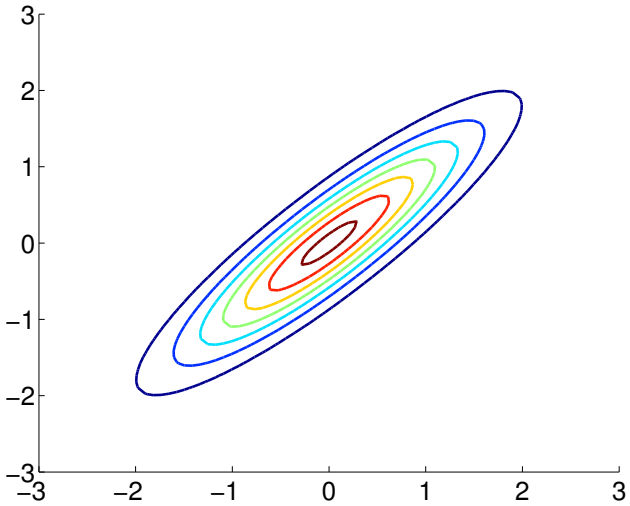

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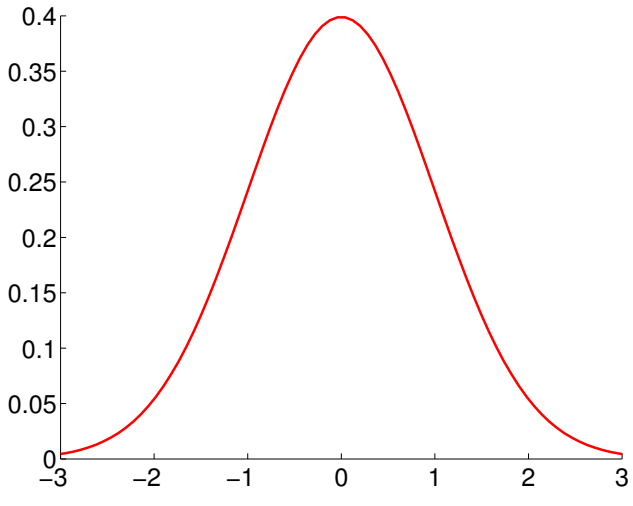

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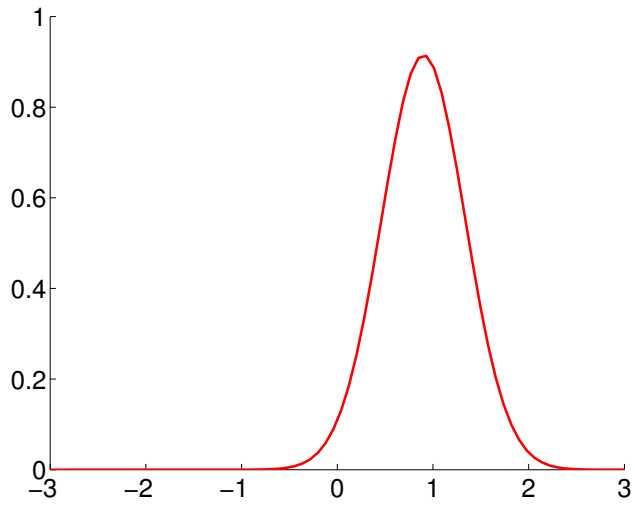

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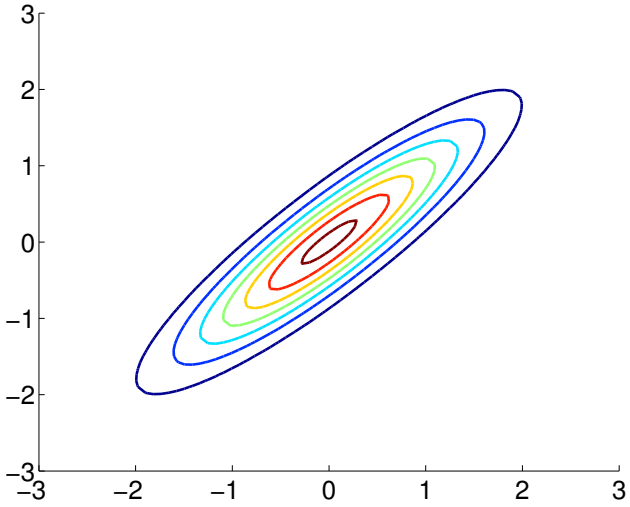

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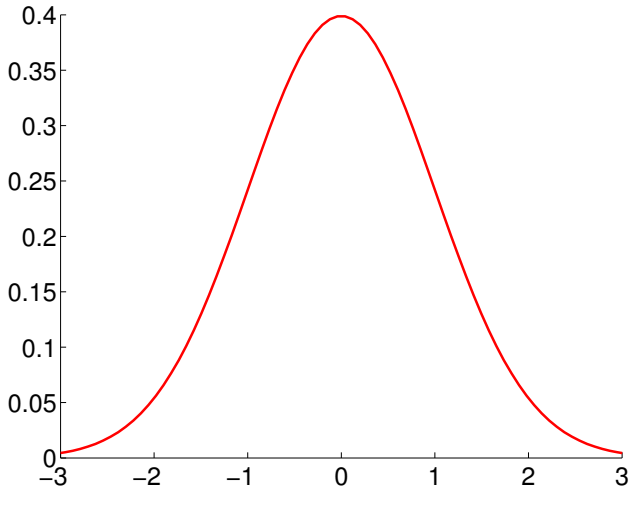

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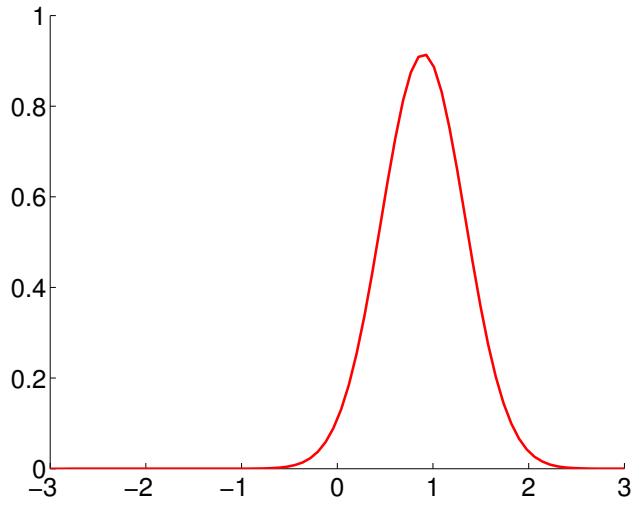

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The Gaussian Distribution

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- An entry $\boldsymbol{\Sigma}_{ij} = 0$ indicates that the variables i and j are marginally independent given all the other variables.

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- An entry $\boldsymbol{\Sigma}_{ij} = 0$ indicates that the variables i and j are marginally independent given all the other variables.
- Marginalizing out a variable leaves $\boldsymbol{\Sigma}$ unchanged but changes $\boldsymbol{\Sigma}^{-1}$.
  - This is crucial when parameterizing a Gaussian process.

Gaussian Quiz

The Standard Linear Regression Model

Notation and Settings

$$\text{Data} : \mathcal{D} = \{(\mathbf{x_i}, \mathbf{y_i})\}_{i=1}^{N},\ \mathbf{x} \in \mathbb{R}^{D},\ \mathbf{y} \in \mathbb{R}$$

$$\text{Input} : (\mathbf{X})_{D \times N},\ \text{Targets:}\ (\mathbf{y})_{N \times 1}$$

$$\text{Goal} : \mathbf{x} \overset{\mathbf{f}(\mathbf{x})}{\rightarrow} \mathbf{y}$$

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\text{Model} \quad & f(\mathbf{x}) = \sum_{i=1}^{D} w_i x_i && = \mathbf{w}^T \mathbf{x} \\
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We need do to inference on $\mathbf{w}$.

Bayesian Linear Regression

Posterior Distribution

Consider a zero-mean Gaussian prior over the weights:

$$\mathbf{w} \sim \mathcal{N}(\mathbf{w}|\mathbf{0}, \mathbf{\Sigma}_w)$$

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- Mean of posterior is equal to its mode
- MAP solution (non-Bayesian): negative log prior as penalty term
- This penalized maximum likelihood is known as *ridge regression*
  - Consider $\mathbf{\Sigma}_w = \lambda\mathbf{I}$ Then :

$$\bar{\mathbf{w}} = (\mathbf{X}\mathbf{X}^{\mathsf{T}} + \frac{1}{\lambda}\sigma^2\mathbf{I})^{-1}\mathbf{X}\mathbf{y}$$

Bayesian Linear Regression

Predictive Distribution

We are interested in making predictions at a new test point $\mathbf{x}_*$

- In fact we obtain the predictive distribution by averaging over all possible parameter values (weighted by their posterior probabilities):

$$\mathrm{p}(\mathrm{f}_*|\mathbf{x}_*, \mathbf{X}, \mathbf{y}) = \int \mathrm{p}(\mathrm{f}_*|\mathbf{x}_*, \mathbf{w})\mathrm{p}(\mathbf{w}|\mathbf{X}, \mathbf{y})\ \mathrm{d}\mathbf{w} = \mathcal{N}(\mathrm{f}_*|\mathbf{x}_*^\mathsf{T}\bar{\mathbf{w}}, \mathbf{x}_*^\mathsf{T}\mathbf{A}^{-1}\mathbf{x}_*)$$

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  - Predictive variance: grows with the magnitude of the test point

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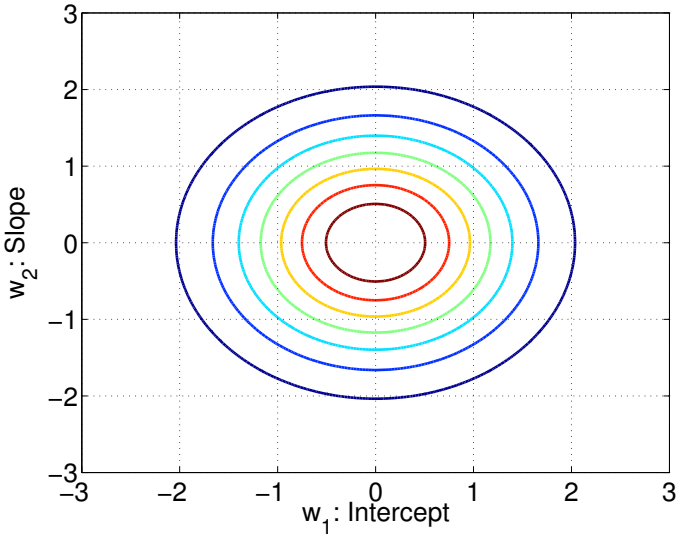
- Point predictions: Need to consider the expected loss (or *risk*):

$$\mathrm{y}_{\mathrm{opt}} = \underset{\mathrm{y}_{\mathrm{pred}}}{\operatorname{argmin}} \int \mathcal{L}(\mathrm{f}_*, \mathrm{y}_{\mathrm{pred}})\mathrm{p}(\mathrm{f}_*|\mathbf{x}_*, \mathbf{X}, \mathbf{y})\mathrm{d}\mathrm{f}_*$$

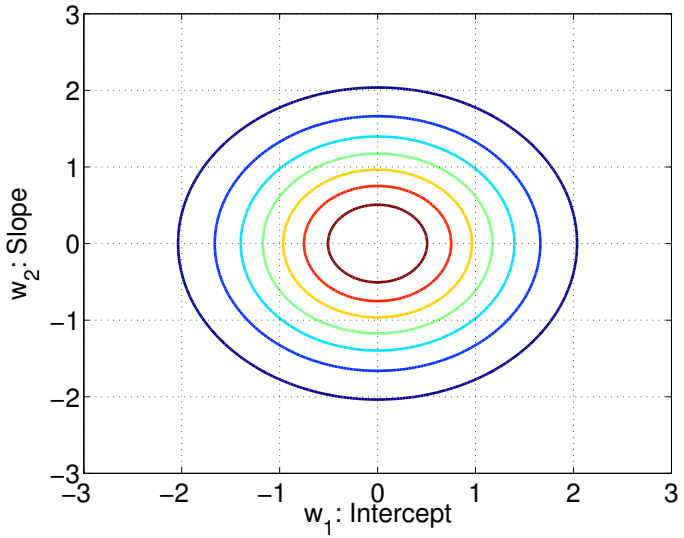
  - e.g. Square loss $\mathcal{L} = (\mathrm{y}_{\mathrm{pred}} - \mathrm{f}_*)^2$
  - c.f. Empirical risk minimization (ERM)

Bayesian Linear Regression Example



Prior Weights

Bayesian Linear Regression Example



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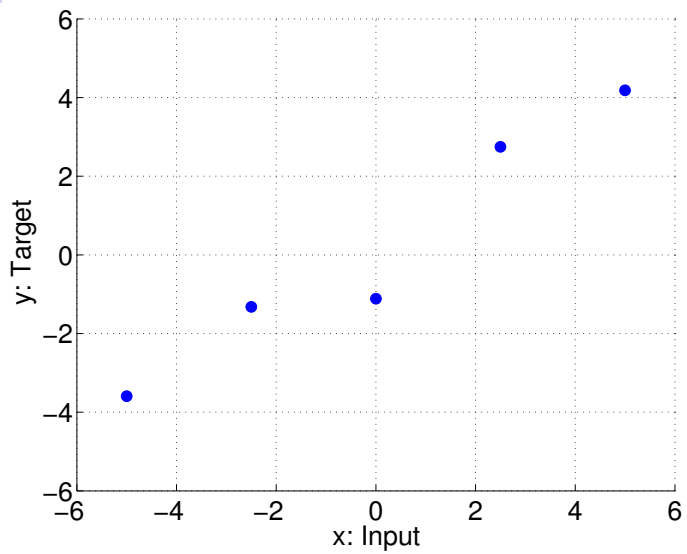
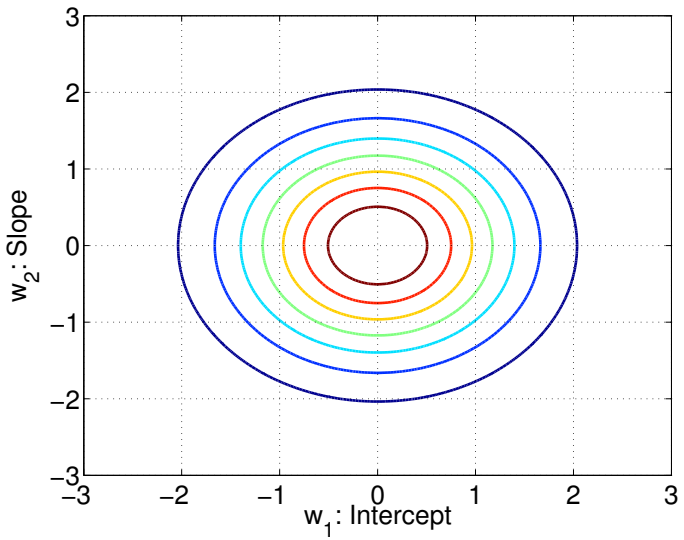


Observed Data

Bayesian Linear Regression Example



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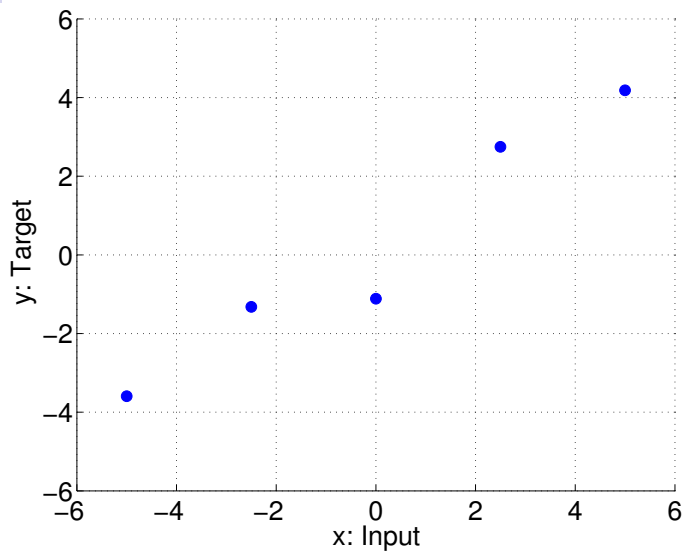

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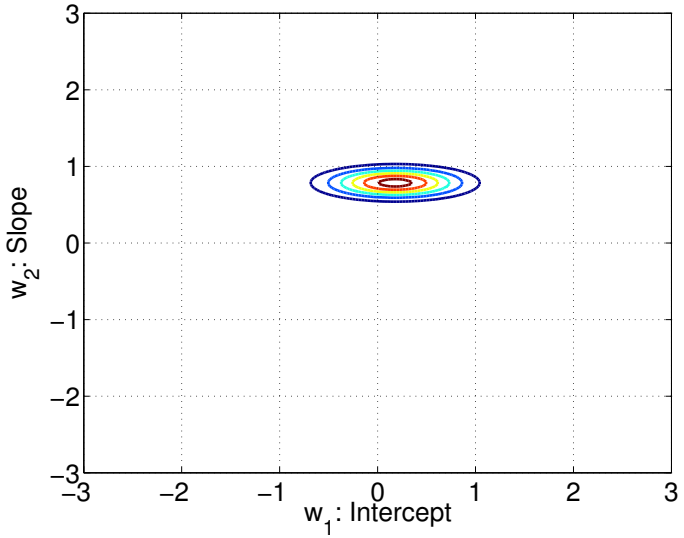
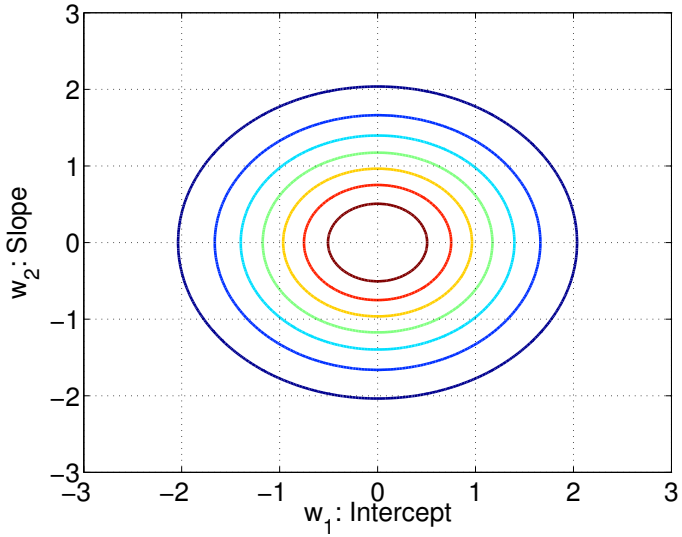


Likelihood

Bayesian Linear Regression Example



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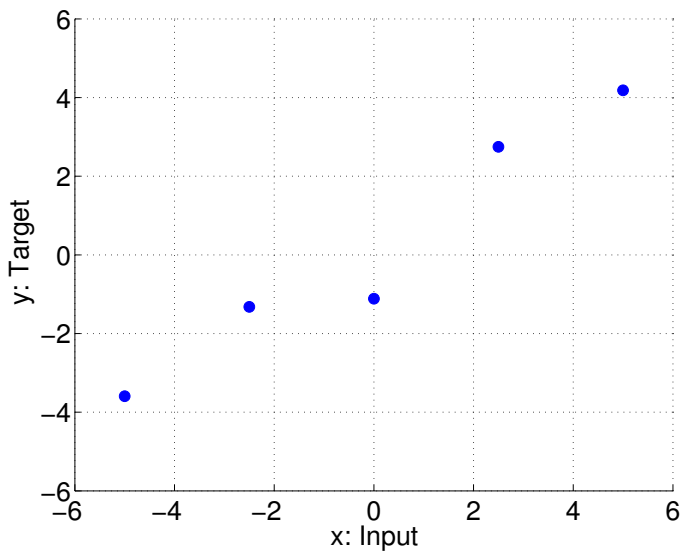

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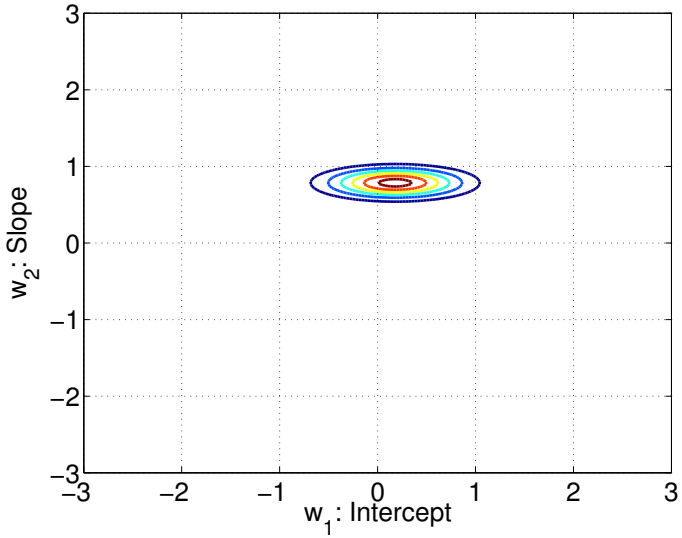

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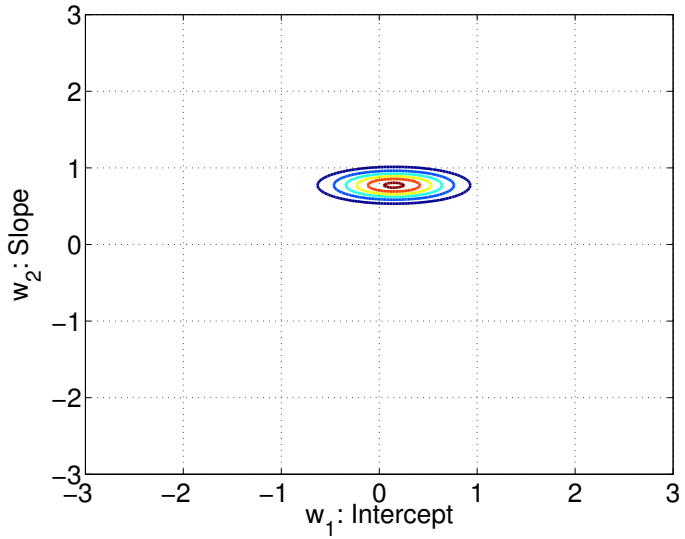
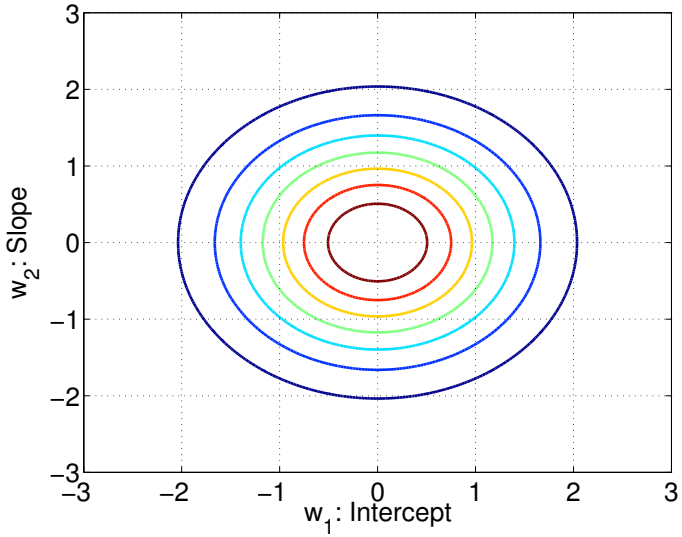


Posterior Weights

Bayesian Linear Regression Example



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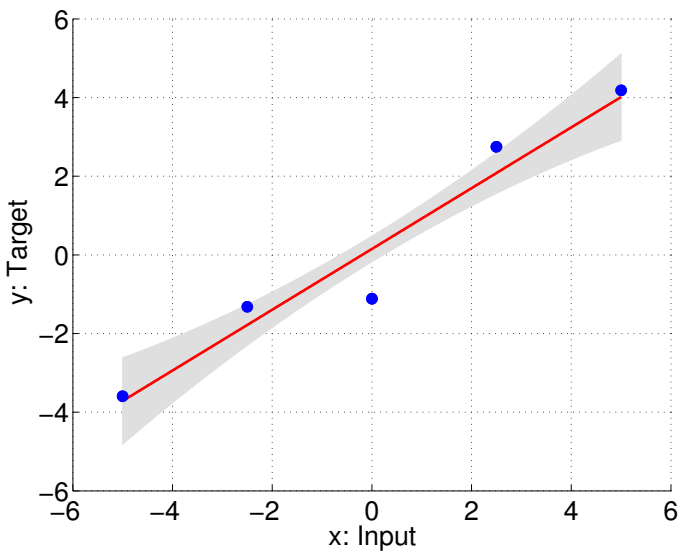
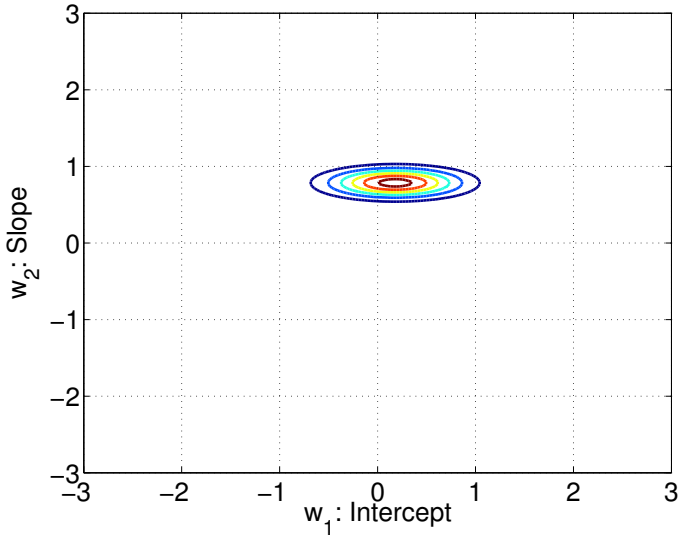


Predictive Distribution



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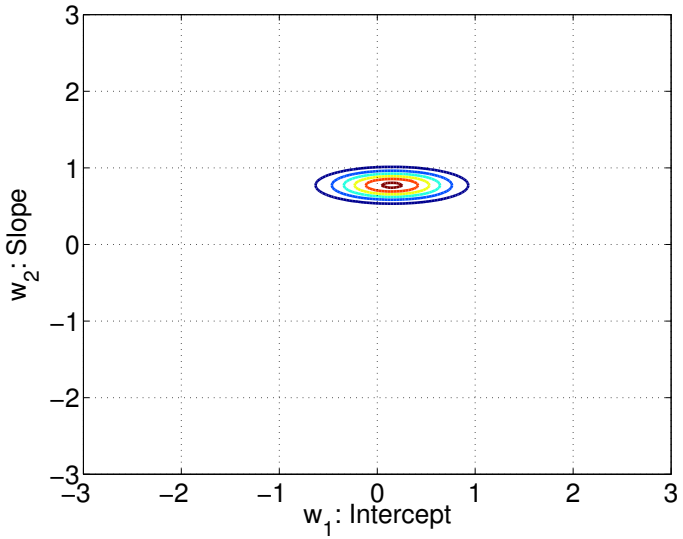

Posterior Weights

Non-linear Feature Spaces

- Consider the model $f(\mathbf{x}) = \sum_{i=1}^{D'} w_i \phi_i(\mathbf{x}) = \mathbf{w}^T \boldsymbol{\phi}(\mathbf{x})$
  - Each $\phi_i(\mathbf{x})$ is a (non-linear) feature on $\mathbf{x}$, e.g. $x_1, x_2, x_1^2, x_2^2, x_1 x_2 \ldots$
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  - Now we need to invert $\widetilde{\mathbf{K}}$ of ? dimensions ▸ GP prediction

The Kernel Trick

- Note that in:

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- $\kappa(\cdot,\cdot)$ is called a kernel or covariance function

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- The Bayesian linear model is a Gaussian process
  - The Function values corresponding to any number of inputs have a joint Gaussian distribution.

Sample Functions from the Linear Model

1. Define $\phi_i(x) = \exp(-\frac{1}{2}(x - \mu_i)^2)$, for $i = 1, 2, 3$
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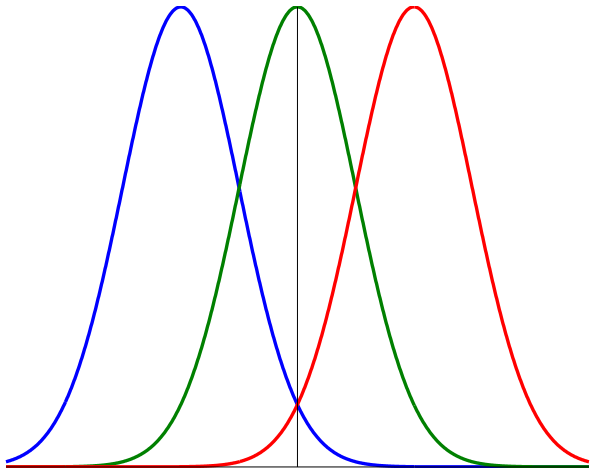

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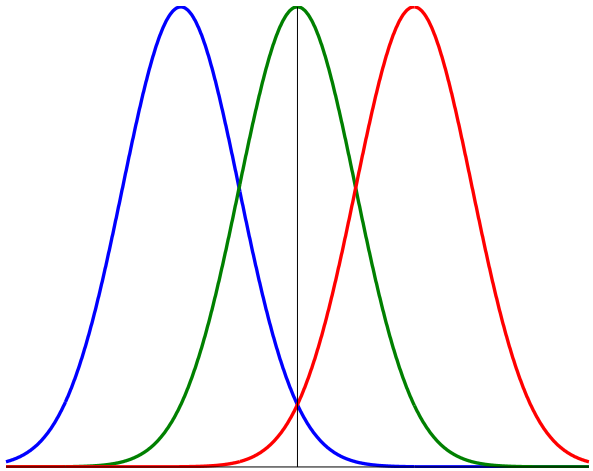

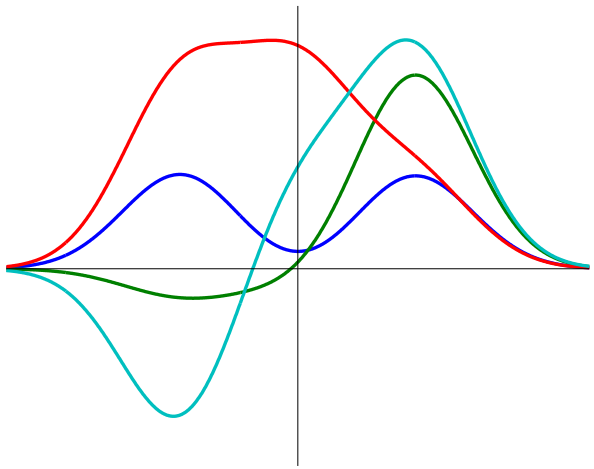

Function-space View

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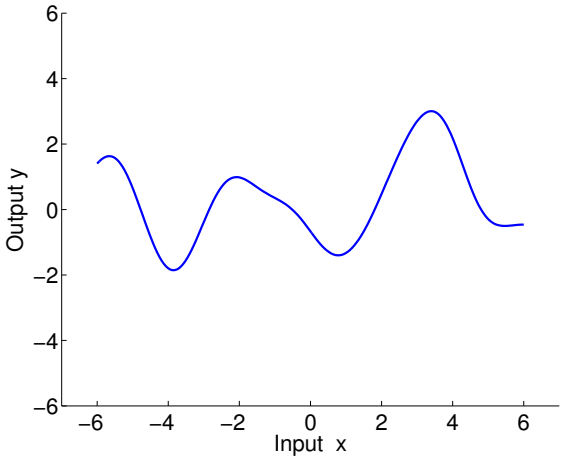
- Each ℓ_j is known as the characteristic length-scale: distance for which the function values are expected to vary significantly

The Squared Exponential (SE) Covariance Function

Example



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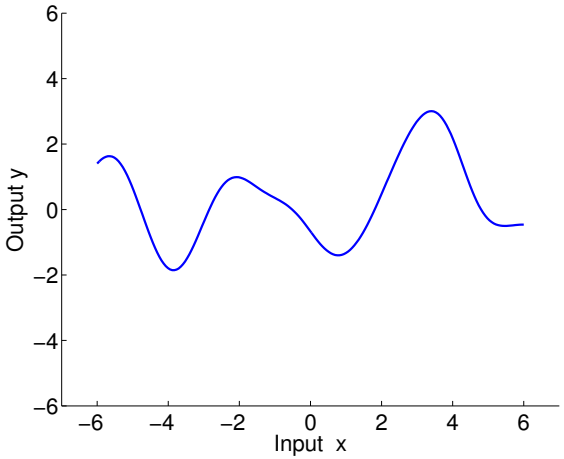

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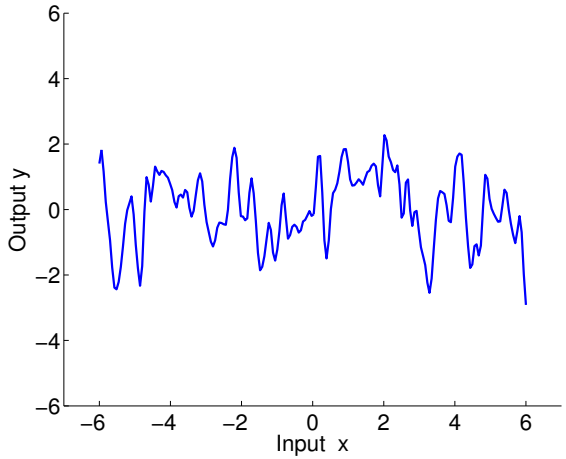

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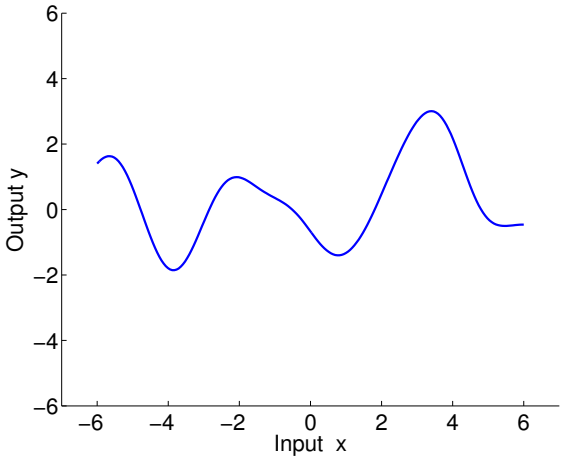

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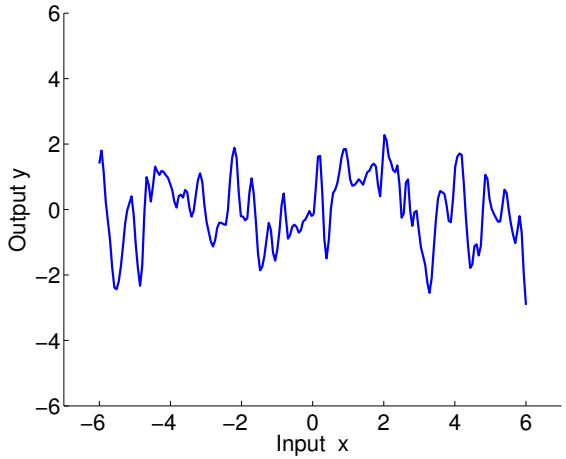

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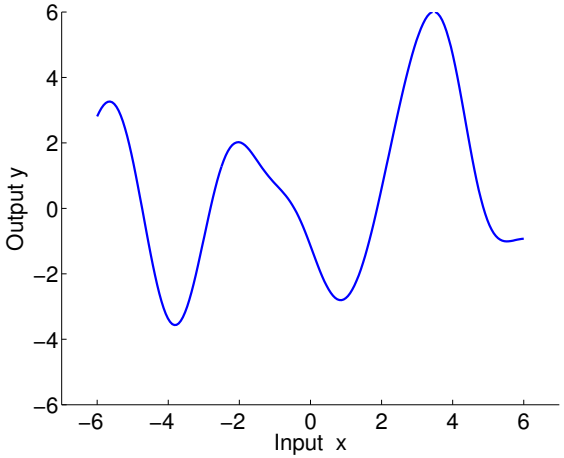

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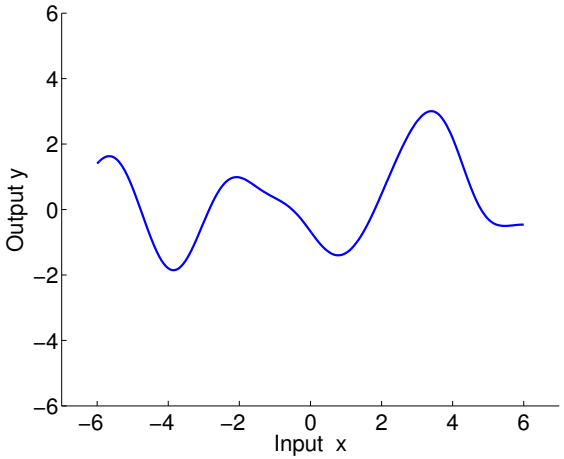

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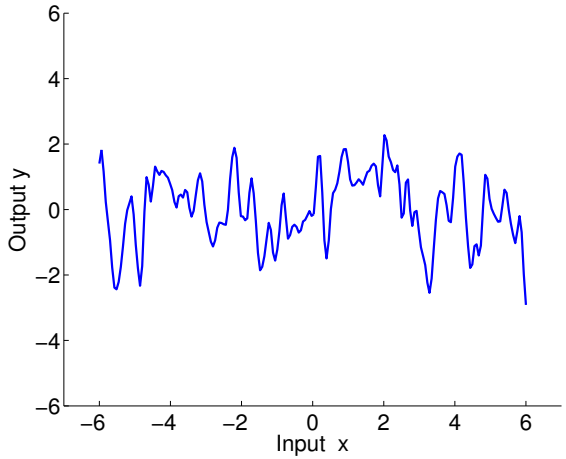

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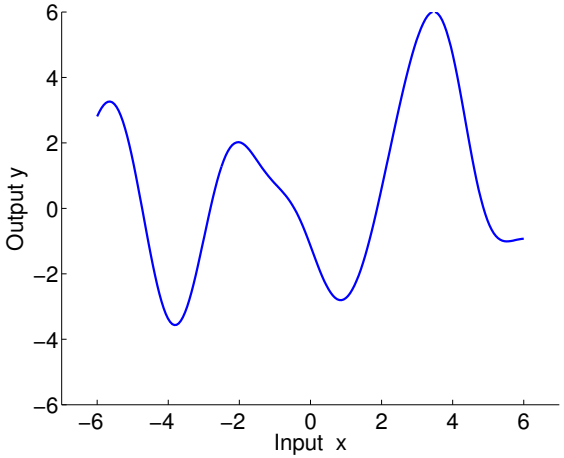

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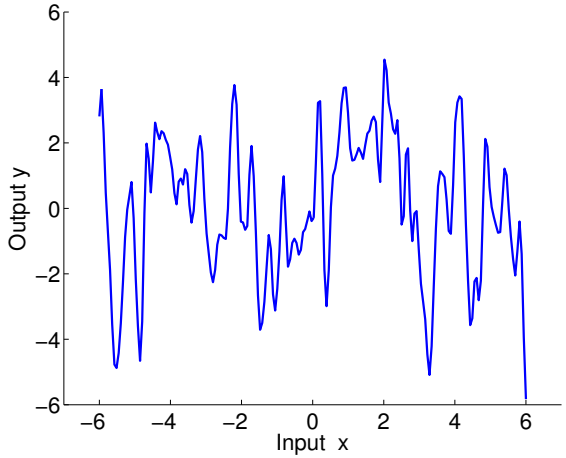

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Data : $\mathcal{D} = \{(\mathbf{x_i}, y_i)\}_{i=1}^{N}$, $\mathbf{x} \in \mathbb{R}^{D}$, $y \in \mathbb{R}$

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$$\text{Data} : \mathcal{D} = \{(\mathbf{x}_i, y_i)\}_{i=1}^{N},\ \mathbf{x} \in \mathbb{R}^D,\ y \in \mathbb{R}$$

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The Graphical Model for GPs

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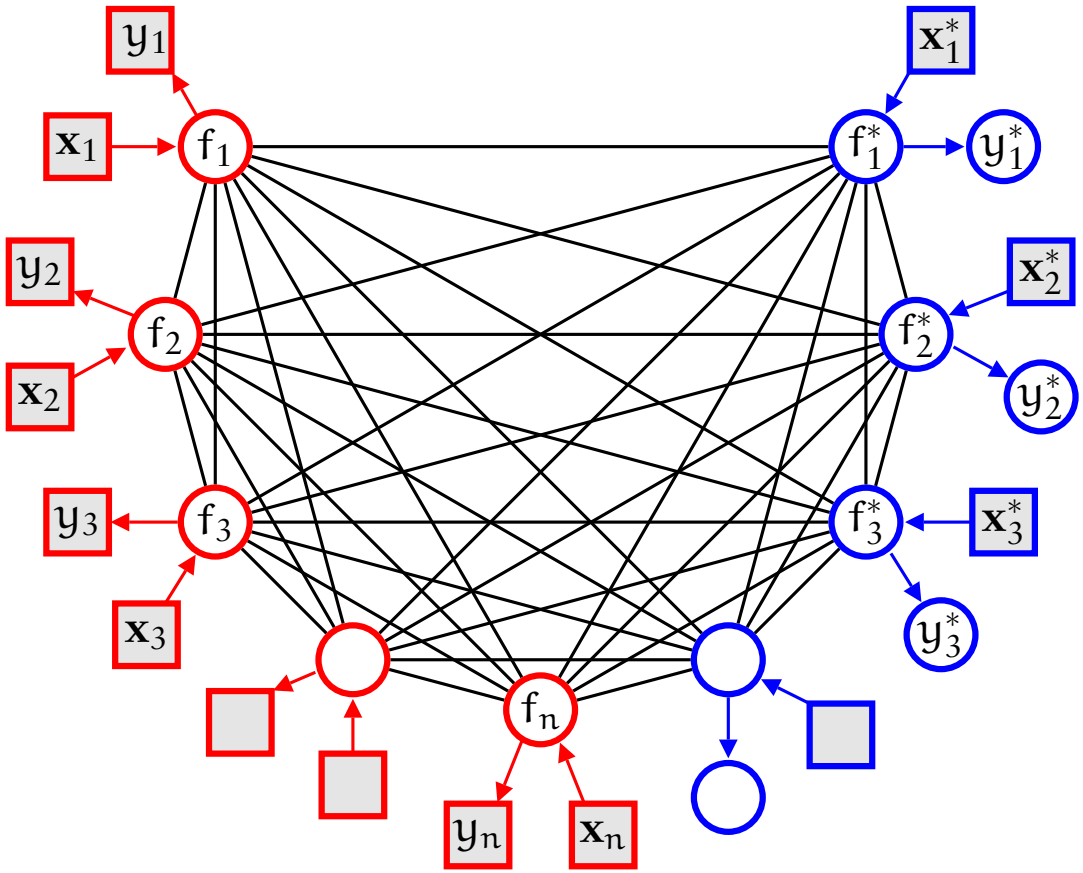

Figure from Carl Rasmussen's slides

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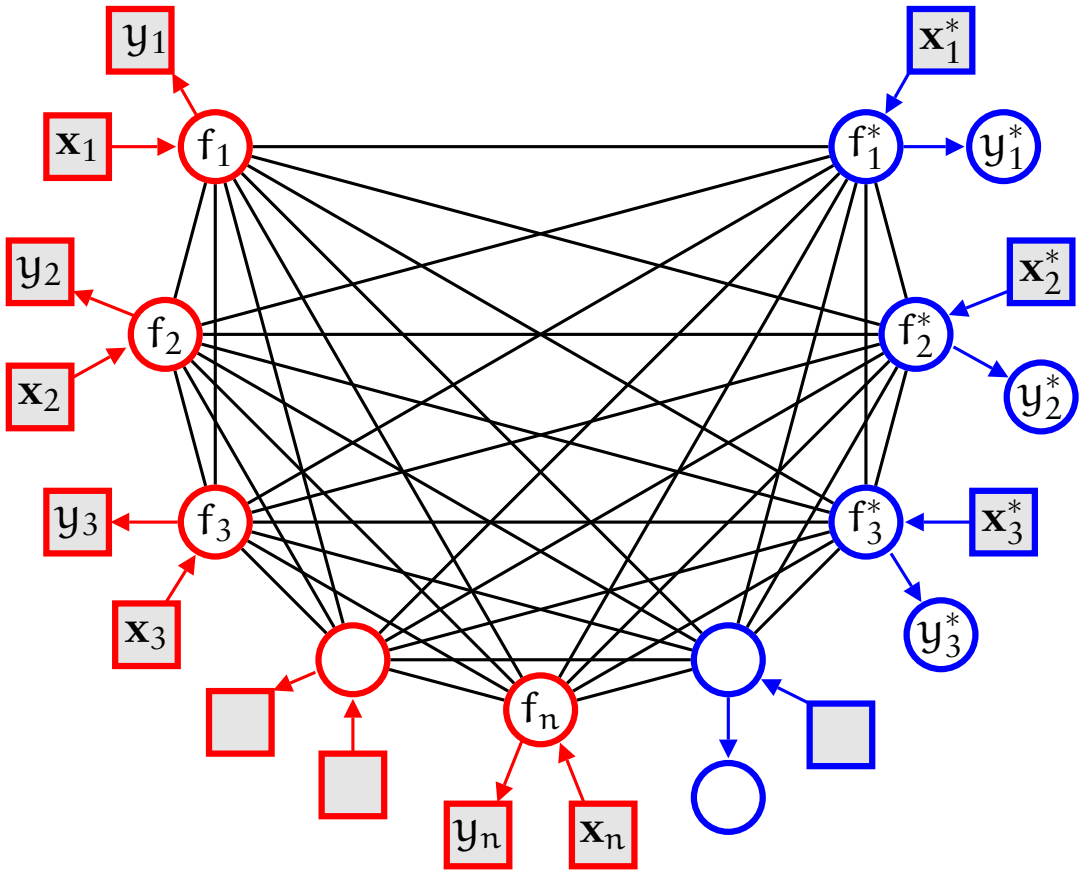

Figure from Carl Rasmussen's slides

- Observations y depend on their corresponding latent function f

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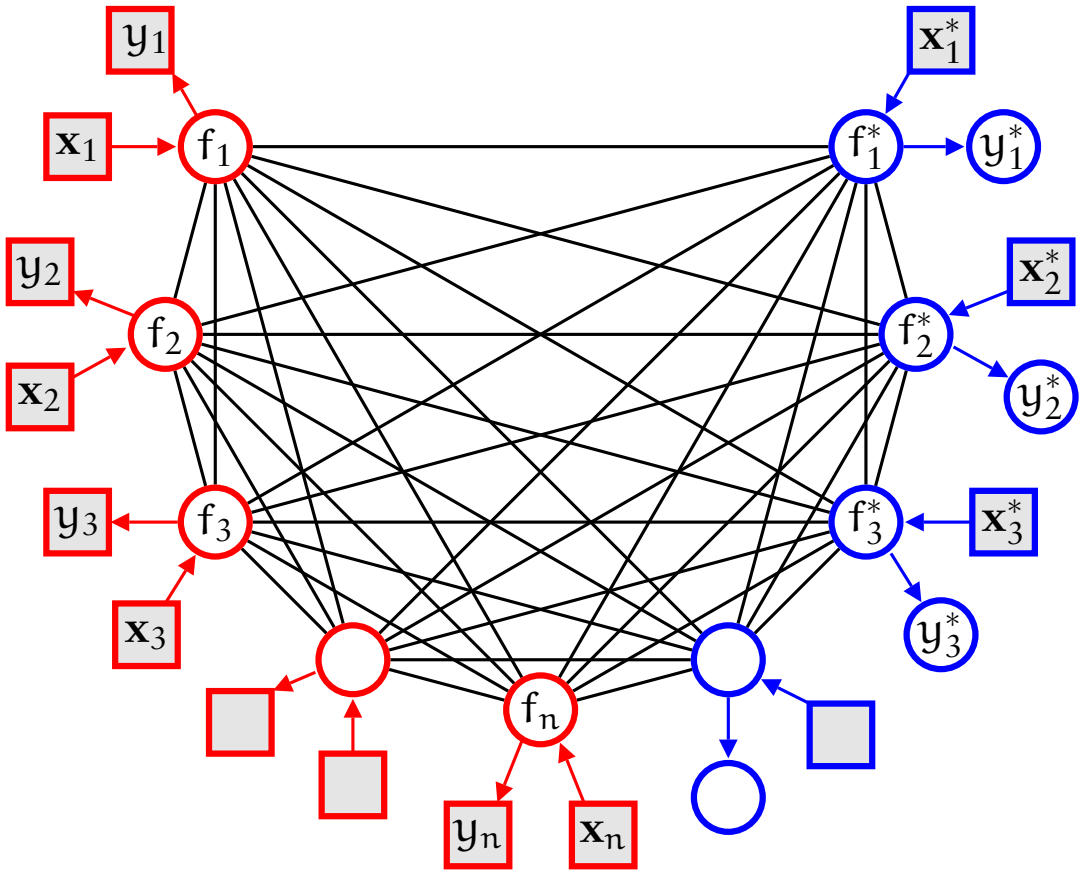

Figure from Carl Rasmussen's slides

- Observations y depend on their corresponding latent function f
- The marginalization property implies that adding a new x_i^*, f_i^*, y_i^* does not affect the distribution

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$$\mathcal{L} = \log p(\mathbf{y}|\mathbf{X}, \theta)$$

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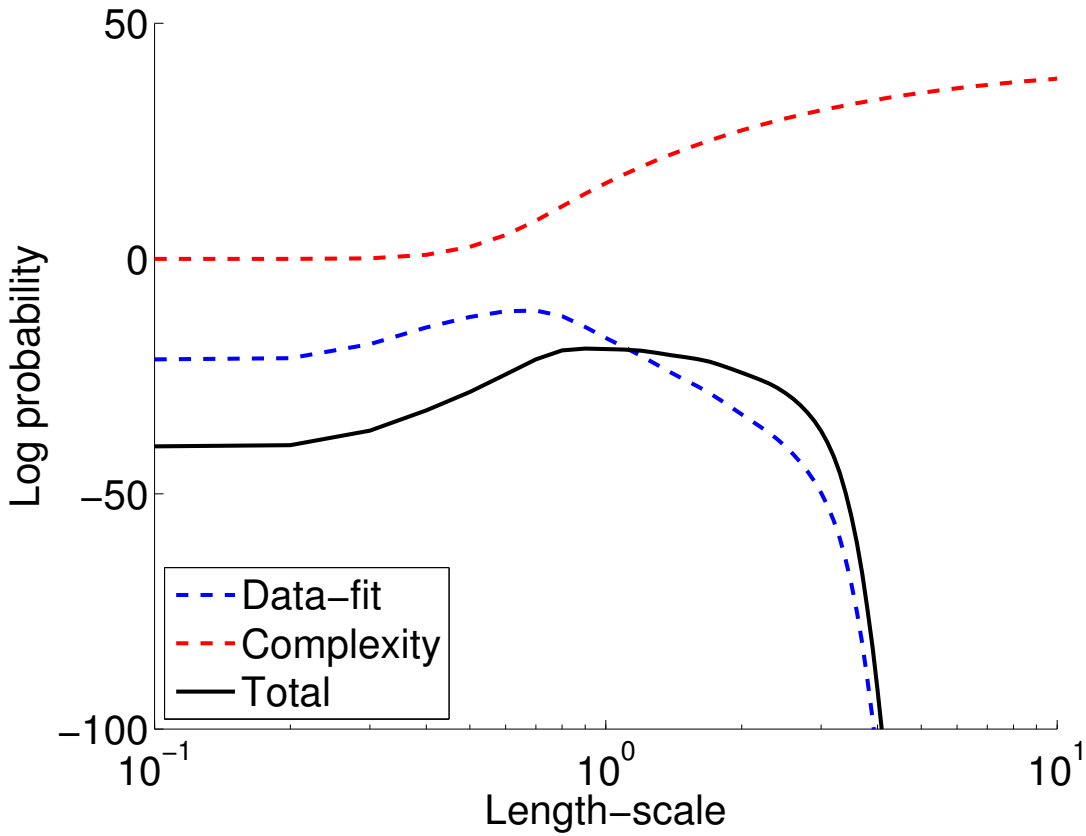

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- Non-convex optimization
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- Computational Requirements?

Automatic Relevance Determination (ARD)

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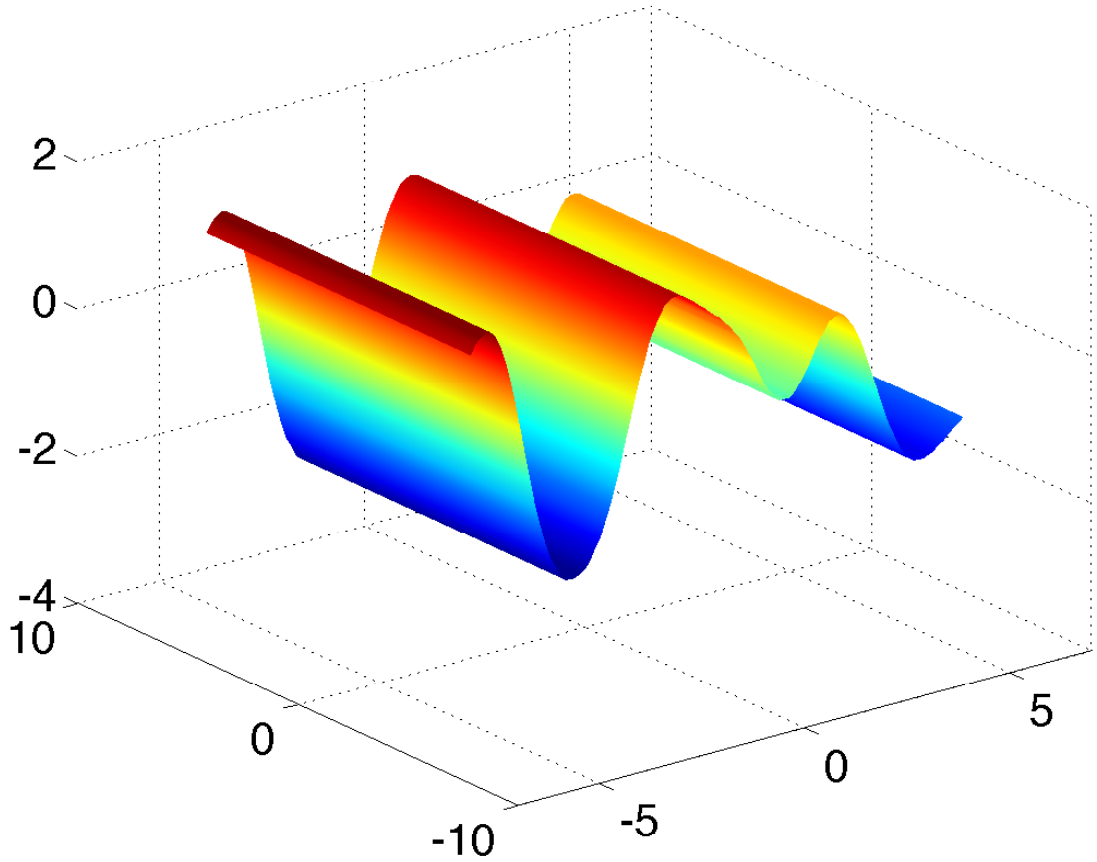

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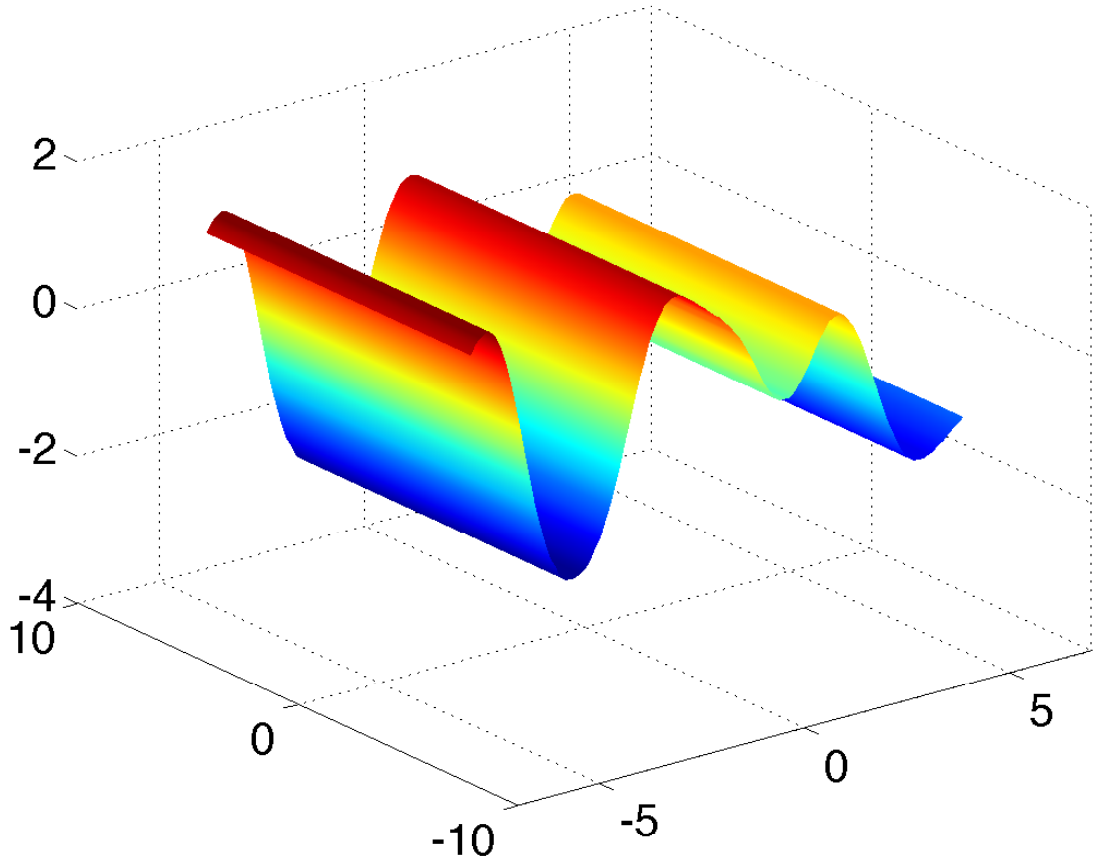

Learned lengh-scale for irrelevant dimension: 1.0557×10^5

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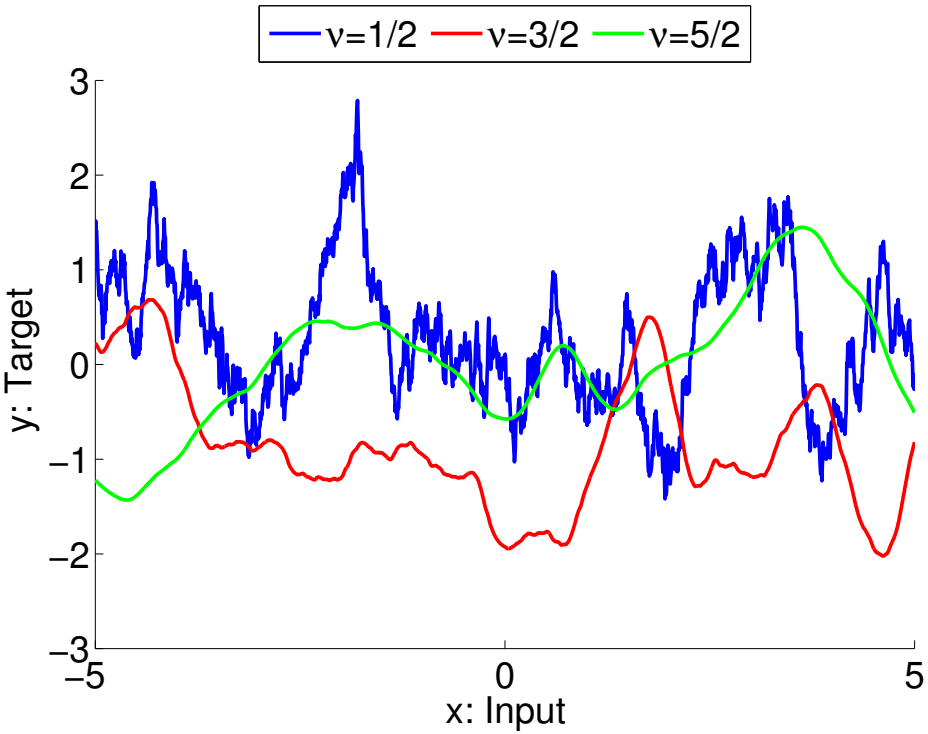

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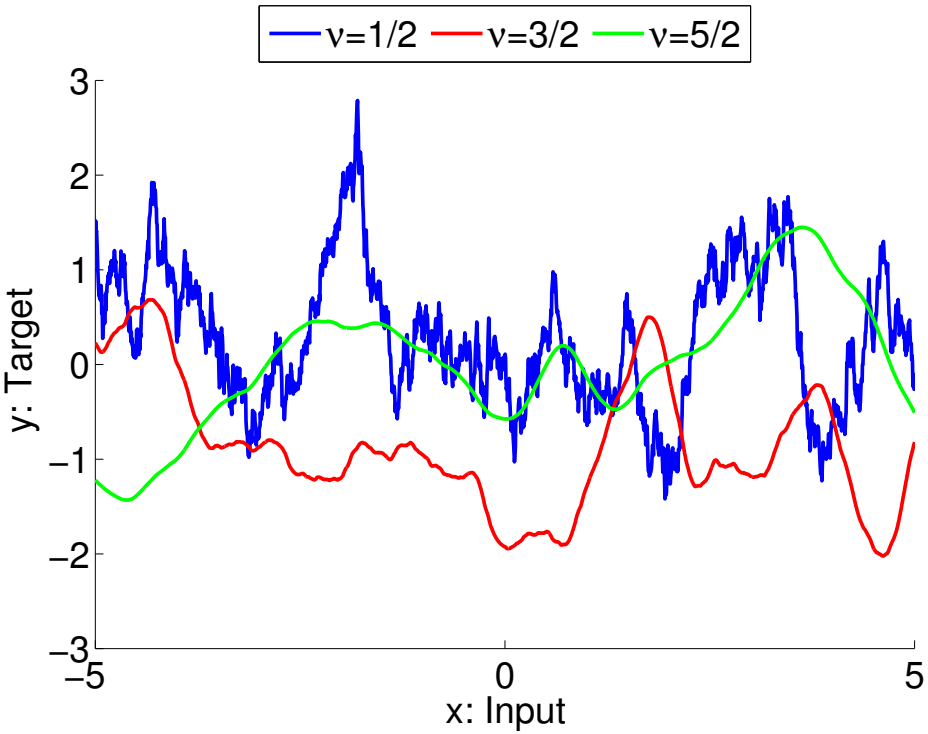

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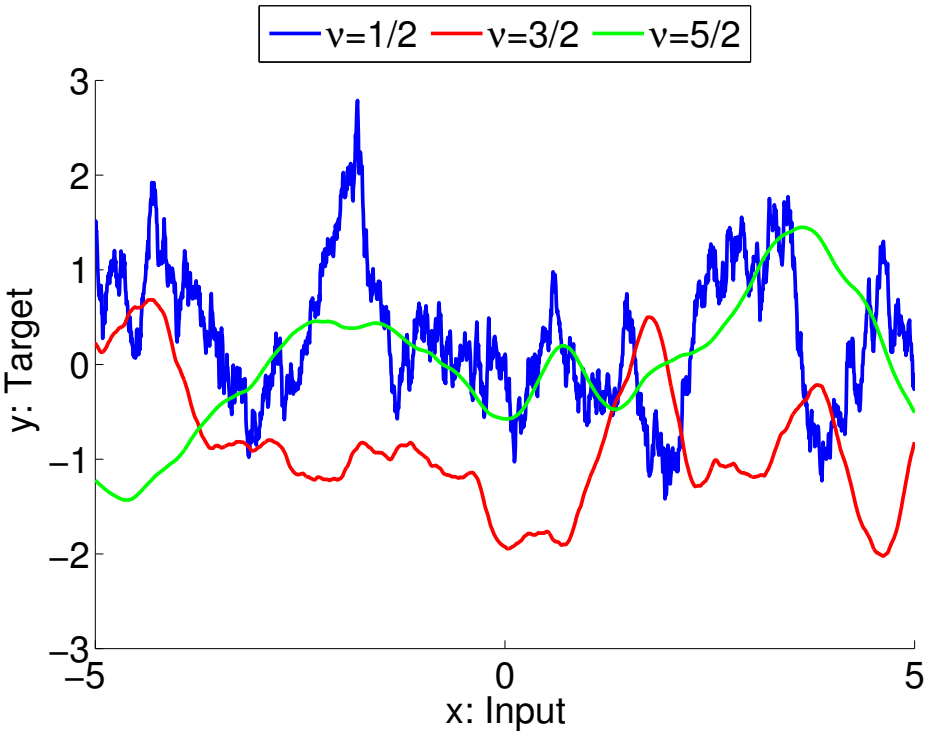

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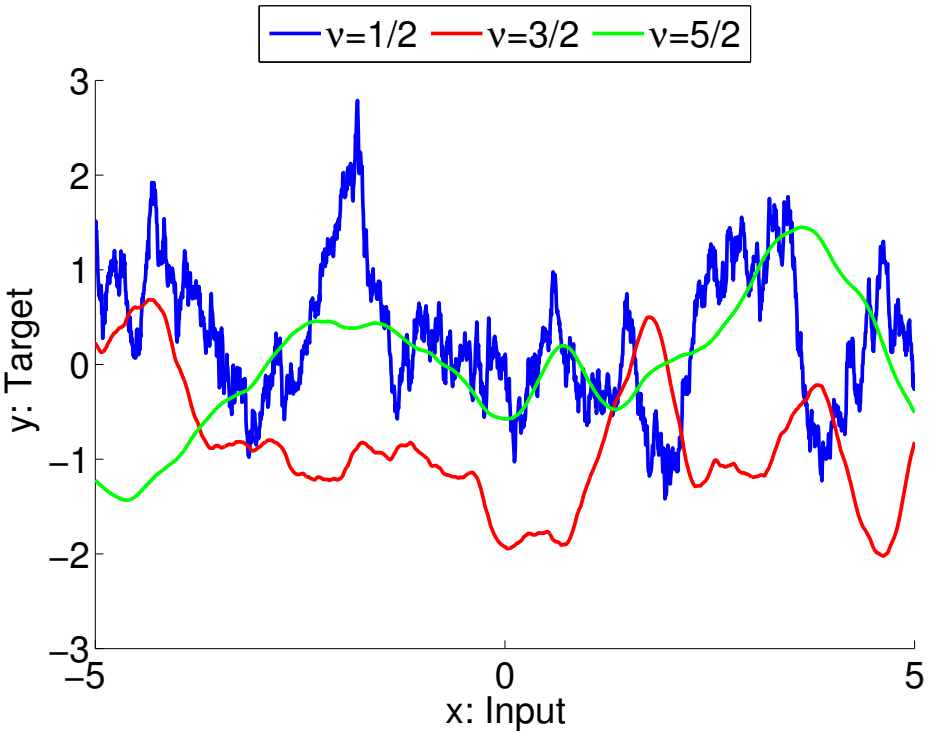

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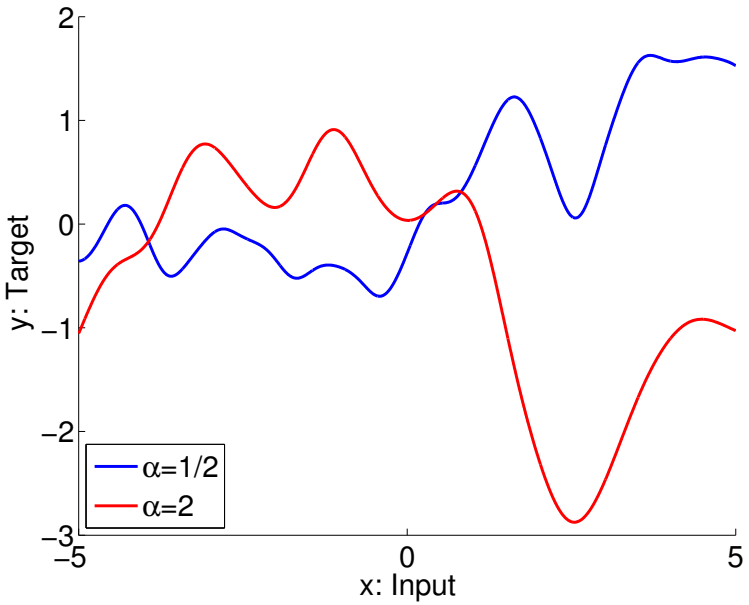

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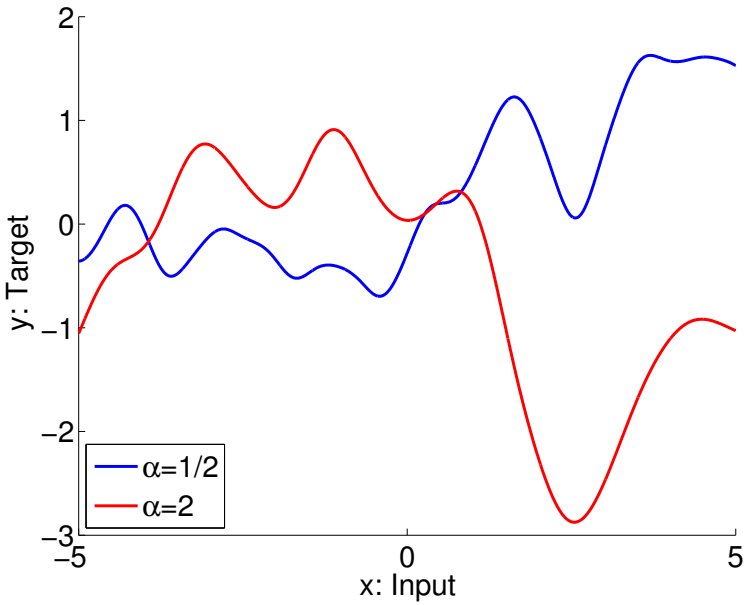
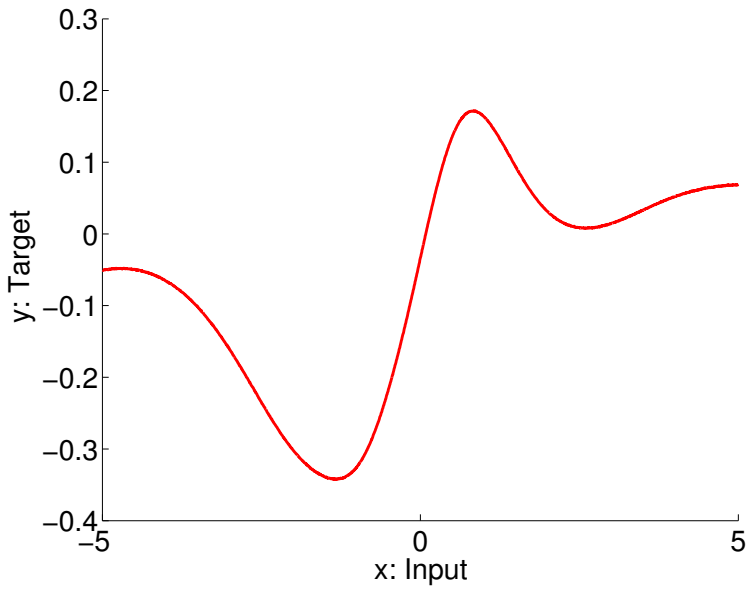


with $\alpha \to \infty$ is the SE covariance with length-scale ℓ.

Other Covariance Functions: Neural Network Covariance

- Consider a neural network with one hidden layer and $\mathrm{N_H}$ hidden units.
- Under certain assumptions the corresponding stochastic process will converge to a Gaussian Process as $\mathrm{N_H} \to \infty$.
- For a specific settings of the transfer function of the neural net:

$$\kappa(\mathbf{x}, \mathbf{x}') = \frac{2}{\pi} \sin^{-1}\left(\frac{2\tilde{\mathbf{x}}^{\mathsf{T}}\boldsymbol{\Sigma}\tilde{\mathbf{x}}'}{\sqrt{(1+2\tilde{\mathbf{x}}^{\mathsf{T}}\boldsymbol{\Sigma}\tilde{\mathbf{x}})(1+2\tilde{\mathbf{x}}'^{\mathsf{T}}\boldsymbol{\Sigma}\tilde{\mathbf{x}}')}}\right)$$



Other Covariance Functions: Periodic, Smooth Functions

We can create a distribution over periodic functions of x by using the mapping $\mathbf{u}(x) = (\cos(x), \sin(x))$ and then use the SE covariance on $\mathbf{u}$ space. This gives rise to:

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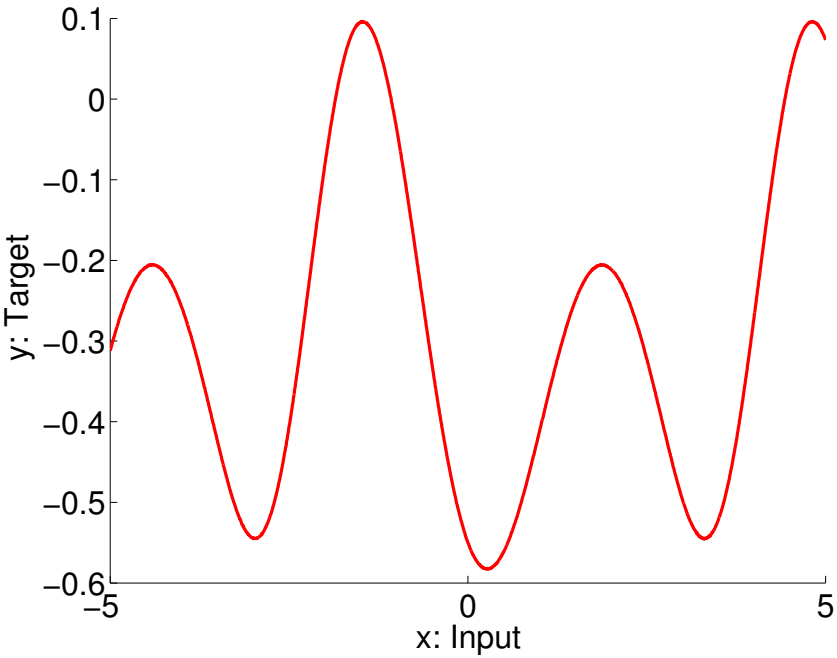

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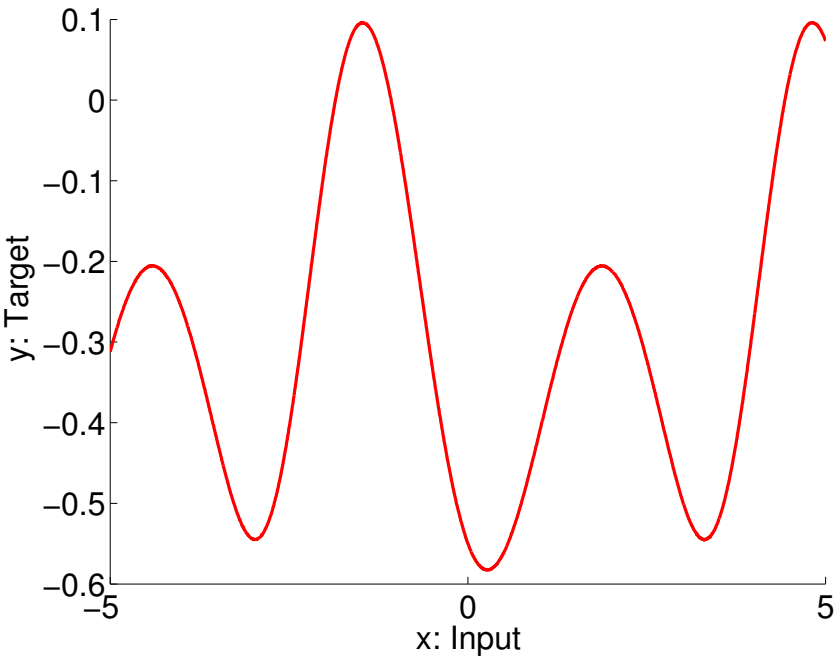

This is called warping and can also be used to introduce non-stationarity.

Gaussian Process Classification: Introduction

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- Generative v discriminative $+$ and $-$?

Linear Models for Classification

Data : $\mathcal{D} = \{(\mathbf{x_i}, y_i)\}_{i=1}^{N}$, $\mathbf{x} \in \mathbb{R}^D$, $y \in \{-1, +1\}$

Input : $(\mathbf{X})_{D \times N}$, Targets: $(\mathbf{y})_{N \times 1}$

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As in Bayesian linear regression we can use the prior:

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Multi-class case is addressed with a softmax function.

Gaussian Process Classification (GPC)

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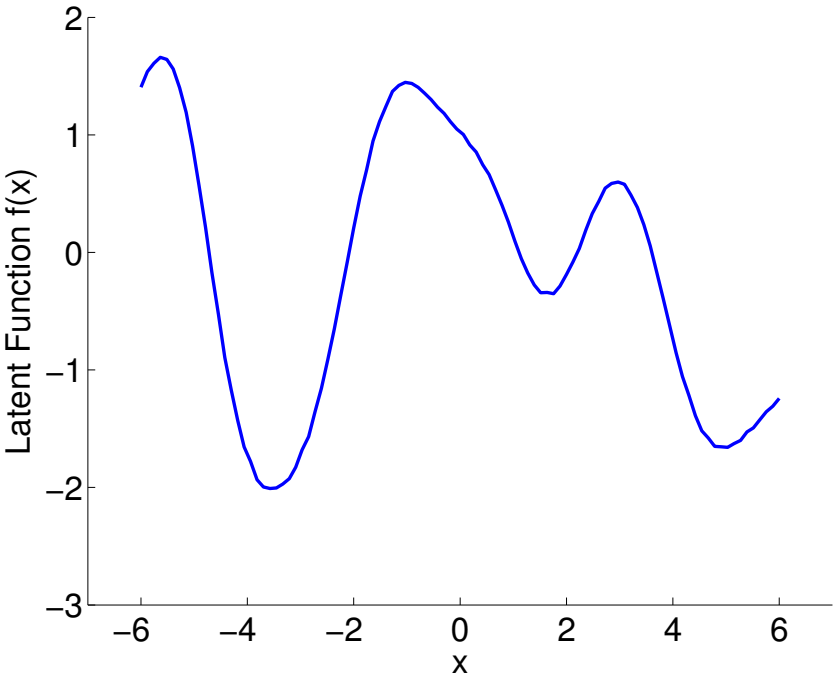

Sample from a GP

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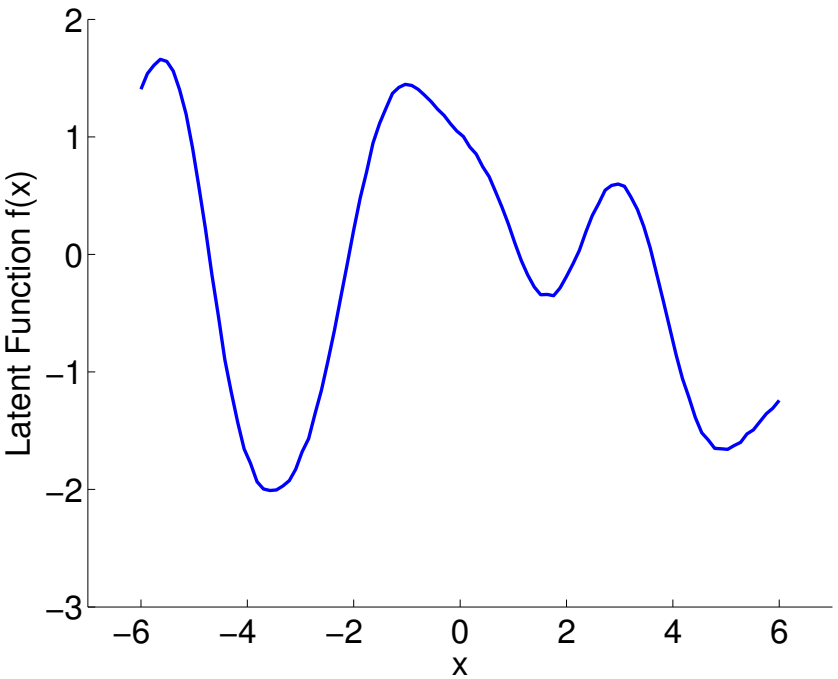

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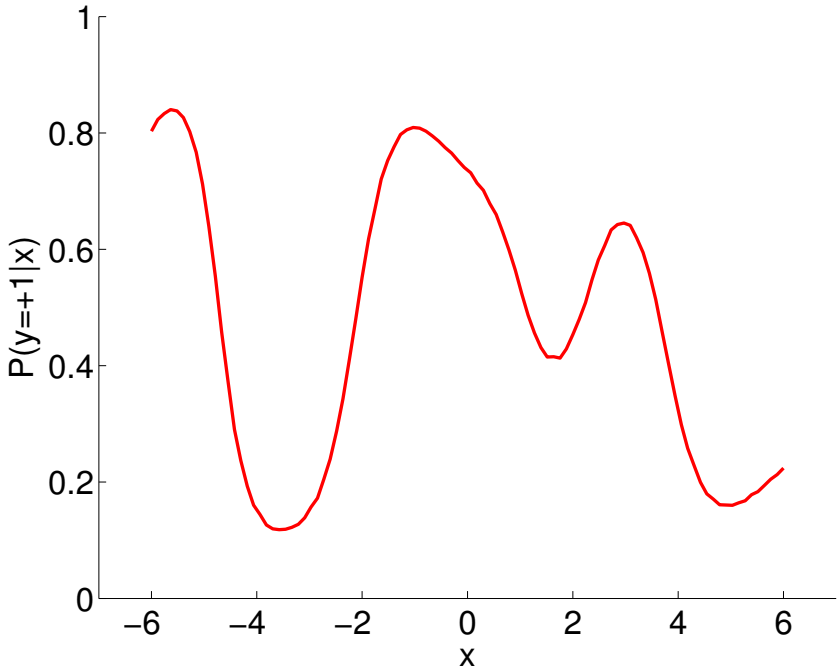

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   - Require numerical approximations (1D) integral for other sigmoid functions

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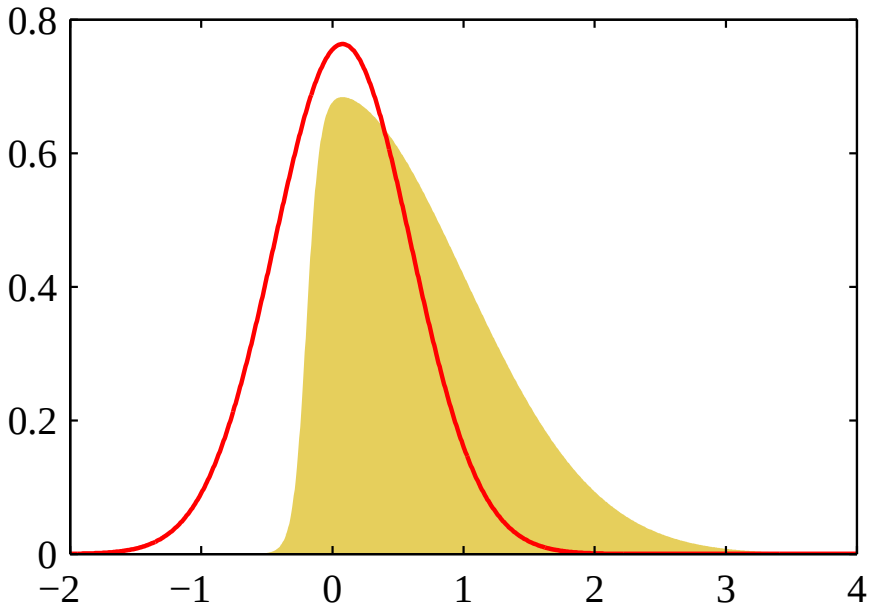

Left : $p(z) \propto \exp(-z^2/2)\sigma(20z + 4)$ and corresponding Gaussian approximation.

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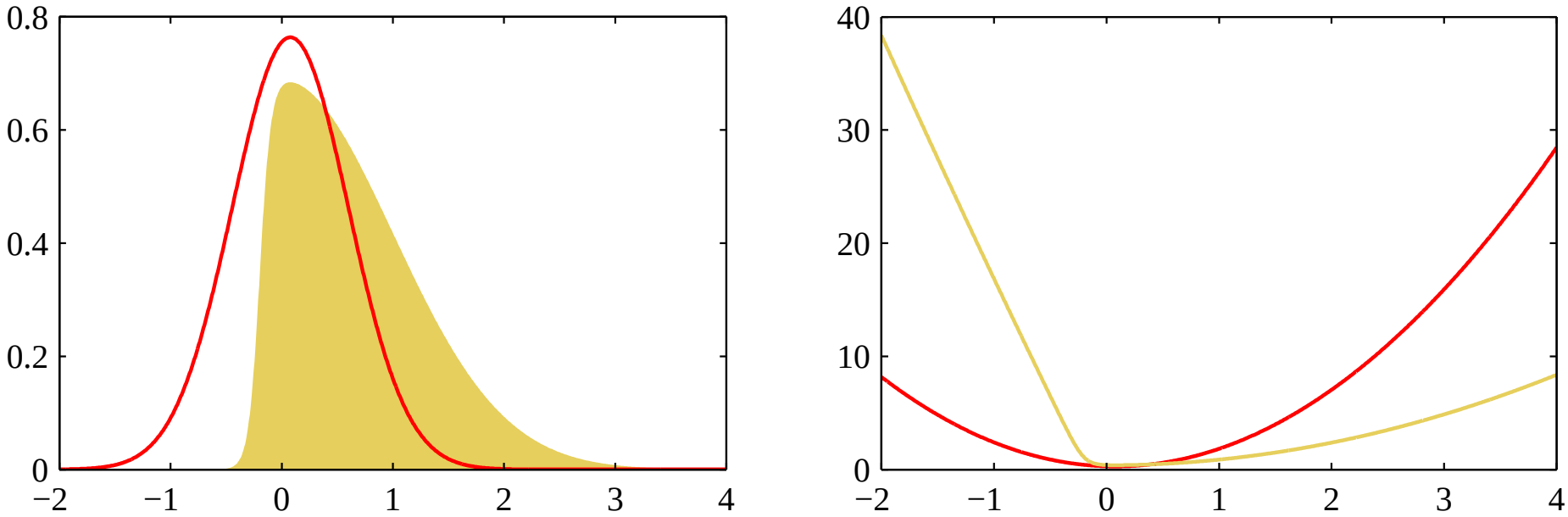

Figures by Christopher M. Bishop (MLPR, 2006)

Left : $p(z) \propto \exp(-z^2/2)\sigma(20z + 4)$ and corresponding Gaussian approximation.

Right : Negative logarithms of the corresponding curves.

The Laplace Approximation to the GP Binary Classifier

Gaussian approximation

$$p(\mathbf{f}|\mathcal{D},\theta) \approx \mathcal{N}(\mathbf{f}|\hat{\mathbf{f}}, A^{-1})$$

where: $\hat{\mathbf{f}} = \text{argmax}_{\mathbf{f}}\, p(\mathbf{f}|\mathcal{D},\theta) = \text{argmax}_{\mathbf{f}}\; p(\mathcal{D}|\mathbf{f},\theta)p(\mathbf{f}|\theta)$ and A is the Hessian of the negative log-posterior evaluated at $\hat{\mathbf{f}}$.

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$$\mathbf{f}^{\text{new}} = (\mathbf{W} + \mathbf{K}^{-1})^{-1}\left(\frac{\partial \log p(\mathbf{y}|\mathbf{f})}{\partial \mathbf{f}} + \mathbf{W}\mathbf{f}\right)$$

$$\text{with } \mathbf{W}_{pq} = \frac{\partial^2 \log p(\mathbf{y}|\mathbf{f})}{\partial f_p \partial f_q}.$$

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$$\mathbf{f}^{\text{new}} = (\mathbf{W} + \mathbf{K}^{-1})^{-1}\left(\frac{\partial \log p(\mathbf{y}|\mathbf{f})}{\partial \mathbf{f}} + \mathbf{W}\mathbf{f}\right)$$

$$\text{with } \mathbf{W}_{pq} = \frac{\partial^2 \log p(\mathbf{y}|\mathbf{f})}{\partial f_p \partial f_q}.$$

Constraint on A? What does this imply?

The Lapace Approximation to GPC

Convergence and Uniqueness:

- Note that $\mathbf{W}$ is a diagonal matrix due to iid assumption
- for concave likelihood functions the un-normalized log posterior has a unique maximum

Once we have found the maximum posterior $\hat{\mathbf{f}}$ by using the above iteration we can show that:

$$\mathrm{p}(\mathbf{f}|\mathcal{D}, \theta) \approx \mathcal{N}(\mathbf{f}|\hat{\mathbf{f}}, (\mathbf{W} + \mathbf{K}^{-1})^{-1}).$$

When is this approximation a good/bad idea?

Posterior and Predictive Distributions

Recalling the posterior distribution:

$$\mathrm{p}(\mathrm{f}_*|\mathbf{X},\mathbf{y},\mathbf{x}_*) = \int \mathrm{p}(\mathrm{f}_*|\mathbf{X},\mathbf{x}_*,\mathbf{f})\mathrm{p}(\mathbf{f}|\mathbf{X},\mathbf{y})\,\mathrm{df}$$

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Hence, under Laplace approximation:

$$\mathbb{E}[\mathrm{f}_*|\mathbf{X}, \mathbf{y}, \mathbf{x}_*] = \mathbf{k}(\mathbf{x}_*)^\mathsf{T}\mathbf{K}^{-1}\hat{\mathbf{f}}$$

$$\mathbb{V}[\mathrm{f}_*|\mathbf{X}, \mathbf{y}, \mathbf{x}_*] = \kappa(\mathbf{x}_*, \mathbf{x}_*) - \mathbf{k}_*^\mathsf{T}(\mathbf{K} + \mathbf{W}^{-1})^{-1}\mathbf{k}_*$$

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For predictions we have two alternatives:

Average $\bar{\pi}_* = p(y_* = +1|\mathbf{X}, \mathbf{y}, \mathbf{x}_*) = \int \sigma(f_*) p(f_*|\mathbf{X}, \mathbf{y}, \mathbf{x}_*) df_*$

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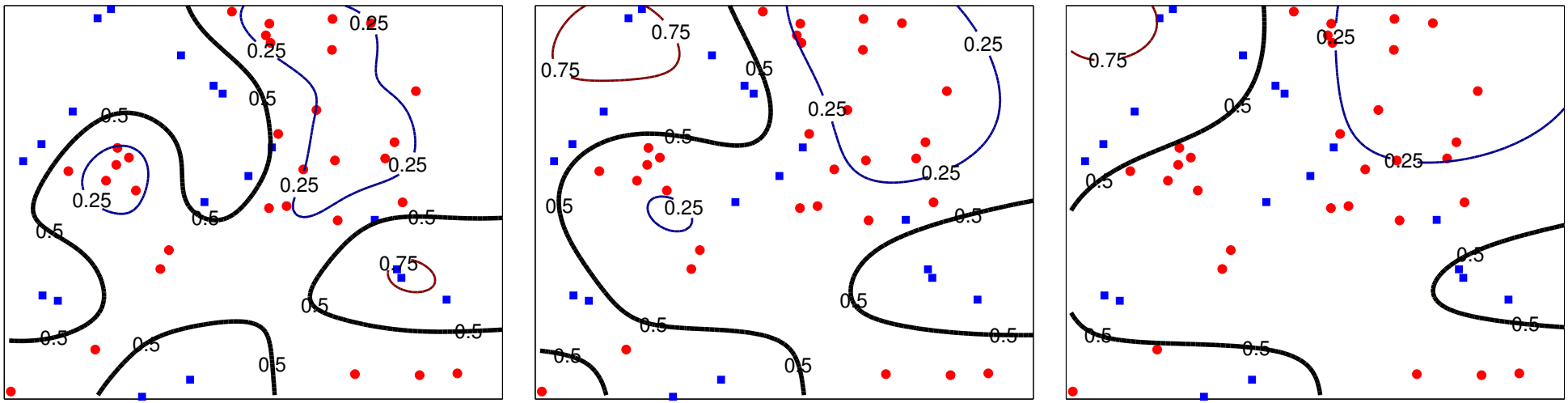
- They provide the same prediction when concerned with most probable classification
- Full distribution is required if we are concerned with confidence in the predictions (e.g. reject options)

Marginal Likelihood and hyper-parameter learning

We can also apply the Laplace approximation to the marginal likelihood:

$$\log p(\mathbf{y}|\mathbf{X}, \theta) \approx -\frac{1}{2}\log|\mathbf{KW} + \mathbf{I}| - \frac{1}{2}\hat{\mathbf{f}}^{\mathsf{T}}\mathbf{K}^{-1}\hat{\mathbf{f}} + \log p(\mathbf{y}|\hat{\mathbf{f}})$$

Predictive probability as a function of the length-scale $\ell = 0.1, 0.2, 0.3$:



Do we spend too much effort in modeling f?

GPR's Computational Complexity

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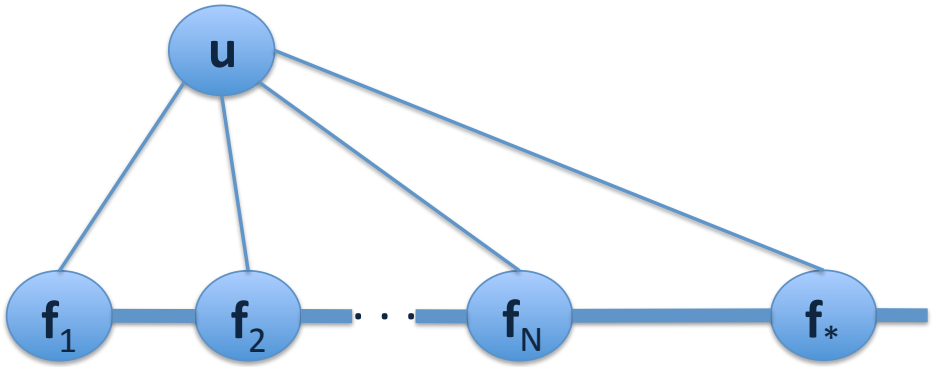
GP Approximations: A Unifying Framework



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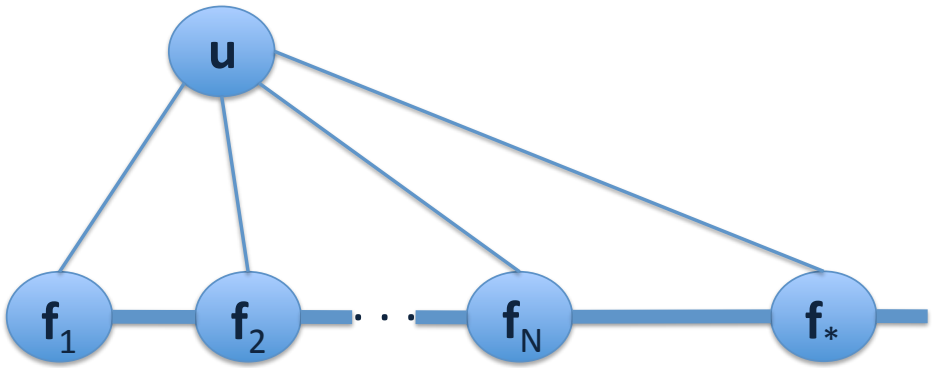

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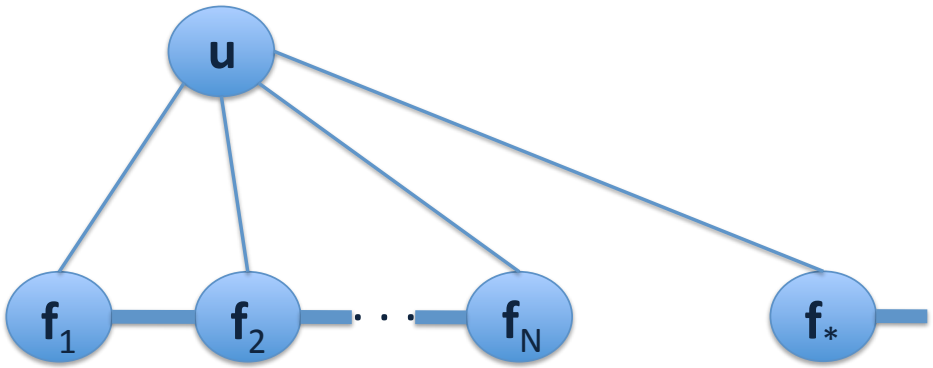

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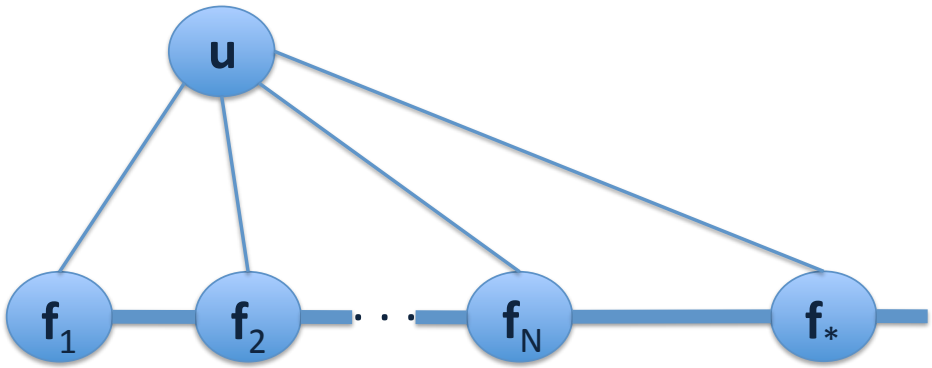

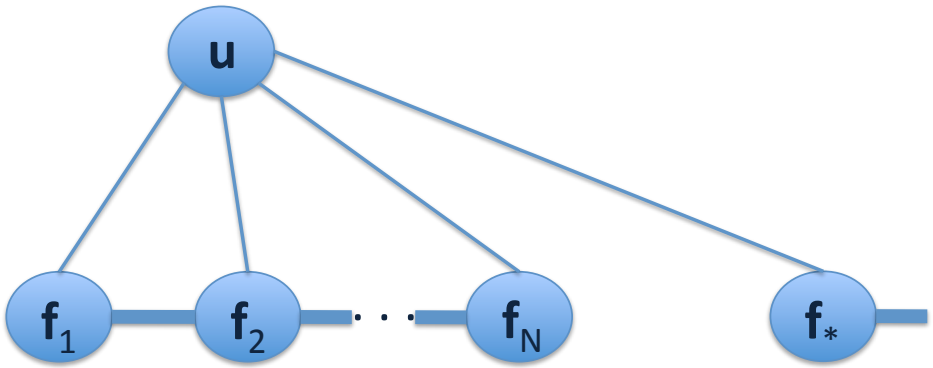

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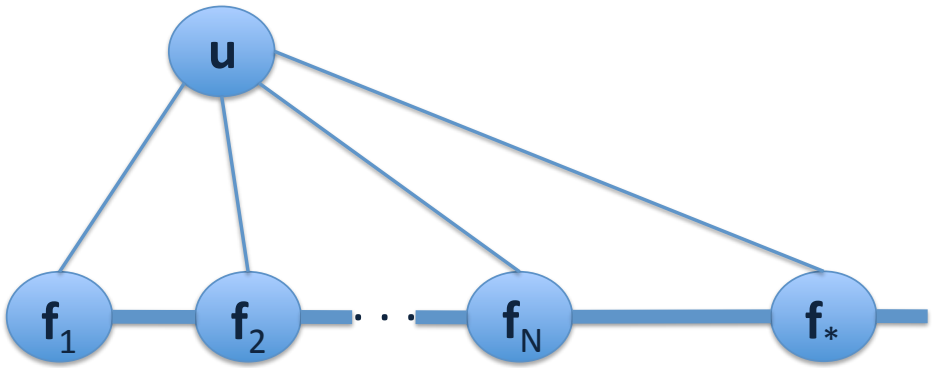

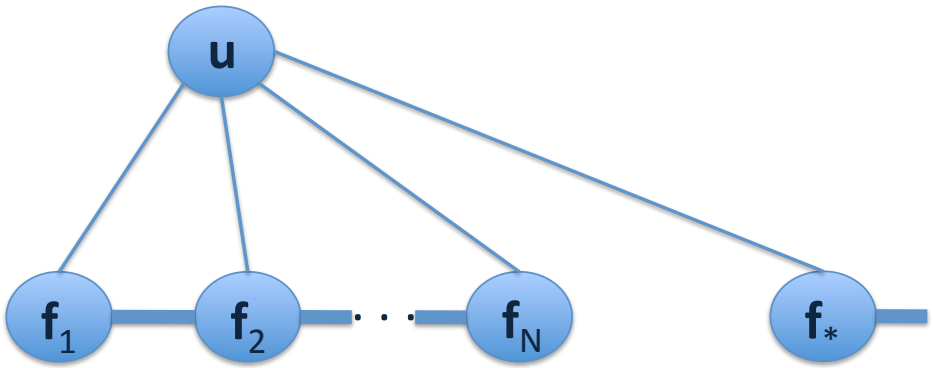

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$\mathrm{q}(\mathbf{f}|\mathbf{u})$ is the training conditional and $\mathrm{q}(\mathbf{f}_*|\mathbf{u})$ is the test conditional.

Most approximation methods can be defined by:

- Different specifications of these conditionals.
- Different $\mathbf{X}_\mathbf{u}$: Subset of training/test points, new $\mathbf{x}$ points

Subset of Regressors (SR)

It can be shown that the mean GP predictor can be obtained by assuming:

$$\text{Prior} : \boldsymbol{\alpha} \sim \mathcal{N}(\mathbf{0}, \mathbf{K}^{-1})$$

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- This method corresponds to a degenerate GP prior
- Complexity: $\mathcal{O}(\tilde{\mathrm{N}}^2\mathrm{N})$ initially and $\mathcal{O}(\tilde{\mathrm{N}})$ and $\mathcal{O}(\tilde{\mathrm{N}}^2)$ per test predictive mean and variance.

Projected Processes (PP)

$$q_{\mathrm{PP}}(\mathbf{f}|\mathbf{u}) = \mathcal{N}(\mathbf{K}_{\mathbf{f},\mathbf{u}}\mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}\mathbf{u}, \mathbf{0}) \quad q_{\mathrm{PP}}(\mathbf{f}_*|\mathbf{u}) = p(\mathbf{f}_*|\mathbf{u})$$

- Inducing variables are a subset of training points
- As in SR, it imposes a deterministic training conditional but (unlike SR) it uses the exact test conditional.
- Same predictive mean as SR but variances are never smaller
- However, this definition implies that the covariances for training cases and test cases are computed differently and therefore this method does not correspond to a (consistent) GP.

FITC, PITC and BCM

FITC : Fully independent training conditionals

PITC : Partially independent training conditionals

BCM : Bayesian Committee Machine

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- PITC uses block diagonal covariance to improve the approximation
- BCM is the same as PITC where the choice of inducing variables depend on the test points, i.e. transductive setting
  - However, note that transduction cannot occur in exact GPs
  - Drawback regarding complexity of transductive models?
  - The choice of $\mathbf{u}$ should not be dictated only by the test points

Sparse GPs (Snelson and Ghahramani, 2006)

- Same as FITC but the inducing inputs do not belong to the training or test sets
- Both the locations of the input points and the values of the hyper-parameters are "learned" by optimization of the approximate marginal likelihood.

GP Approximations: Final Remarks

- The order of computational complexity is identical for all methods (except SD)
- Hence, there is no "excuse for gross approximations"
- Inconclusive experiments on real datasets (See e.g. Rassmussen and Williams, 2006)
- Similar methods for GP classification but we also need to deal with non-Gaussian likelihoods (e.g. using Laplace)
  - Derivatives of the marginal likelihood can get complicated

Multi-task Learning (MTL)

- General idea:
  - Sharing information across tasks (Caruana, 1997)
  - Very little data on test task
  - Exam score prediction, compiler performance prediction, robot inverse dynamics, multi-topic text categorisation, collaborative filtering, multi-level modelling

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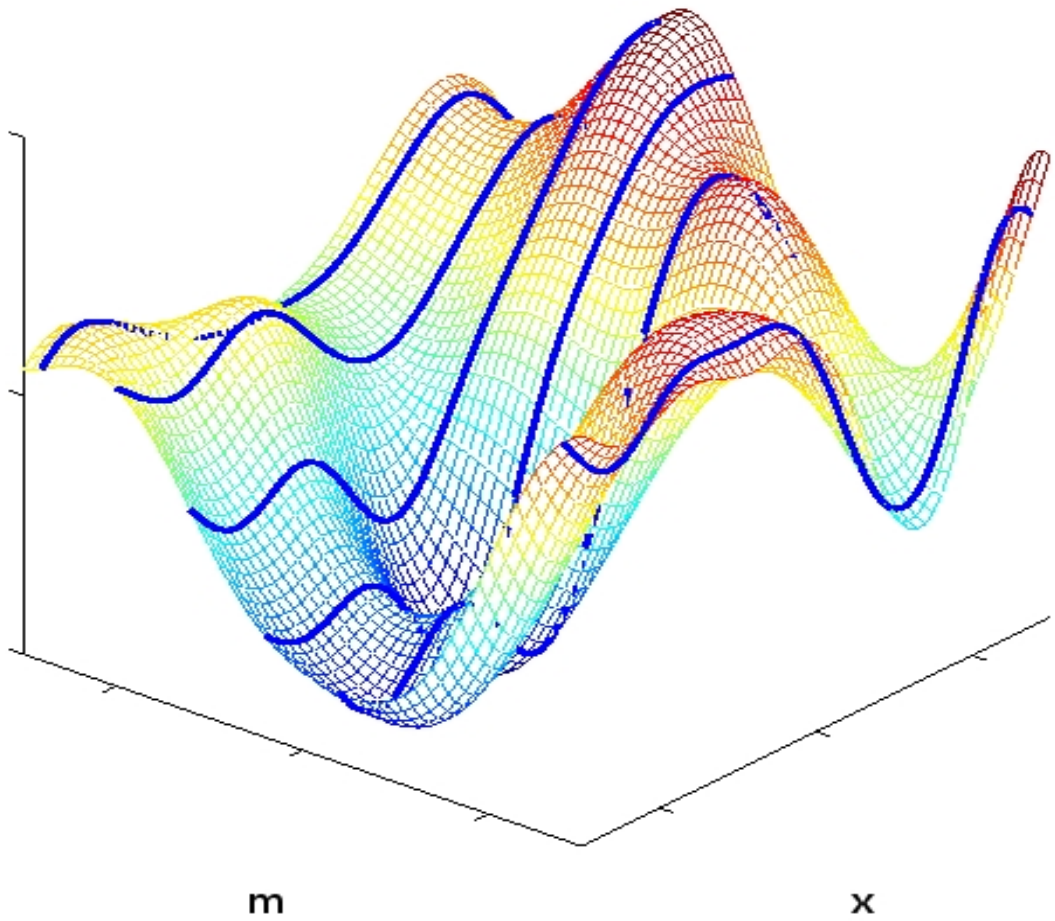
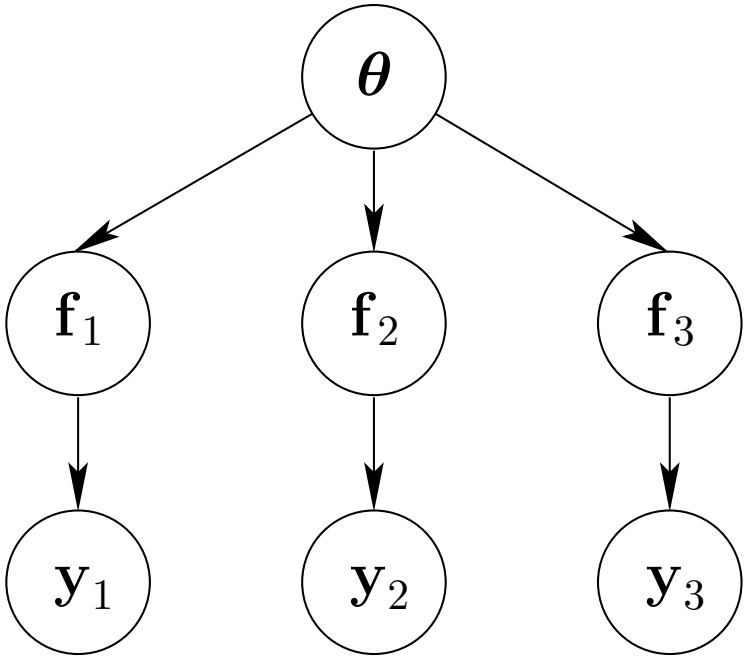
  - Very little data on test task
  - Exam score prediction, compiler performance prediction, robot inverse dynamics, multi-topic text categorisation, collaborative filtering, multi-level modelling
- Assuming task relatedness can be detrimental (Caruana, 1997; Baxter, 2000)
- Task descriptors may be available (Bonilla et al, AISTATS 2007)
- Tasks descriptors unavailable or difficult to define correctly (Bonilla et al, NIPS 2008)
  - e.g. Compiler performance prediction: code features, responses

Multi-task GP: Illustration



Sample functions for different values of tasks (on m axis) are correlated (cf *independent* draws over sample functions)

Inter-task Tying by Hyper-parameter Sharing



Other approaches

- Block diagonal covariance matrix, and each of the M blocks is induced from the same kernel function (Minka and Picard, 1999; Lawrence and Platt, 2004; Yu et al, 2005; Schwaighofer et al, 2005)

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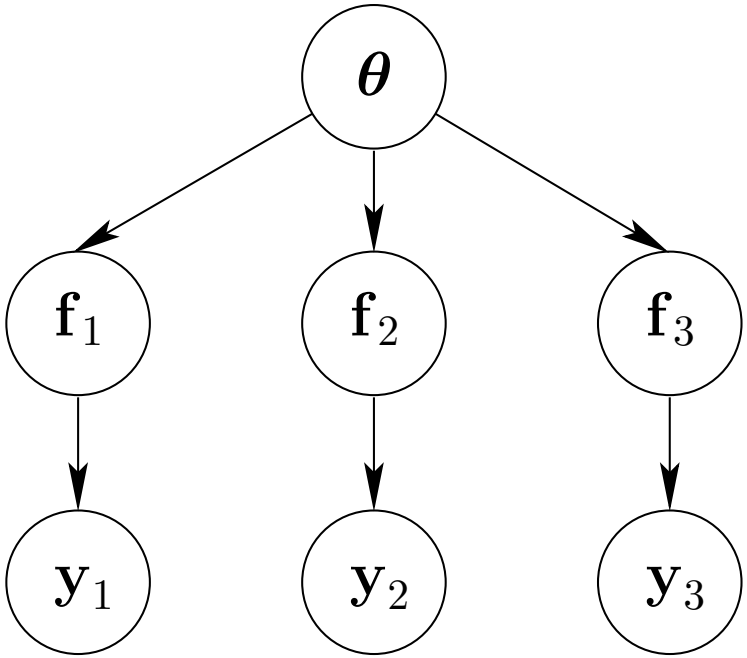

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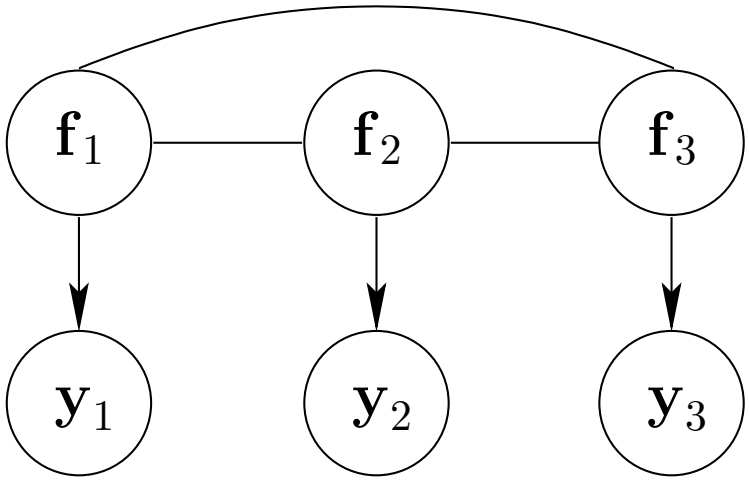

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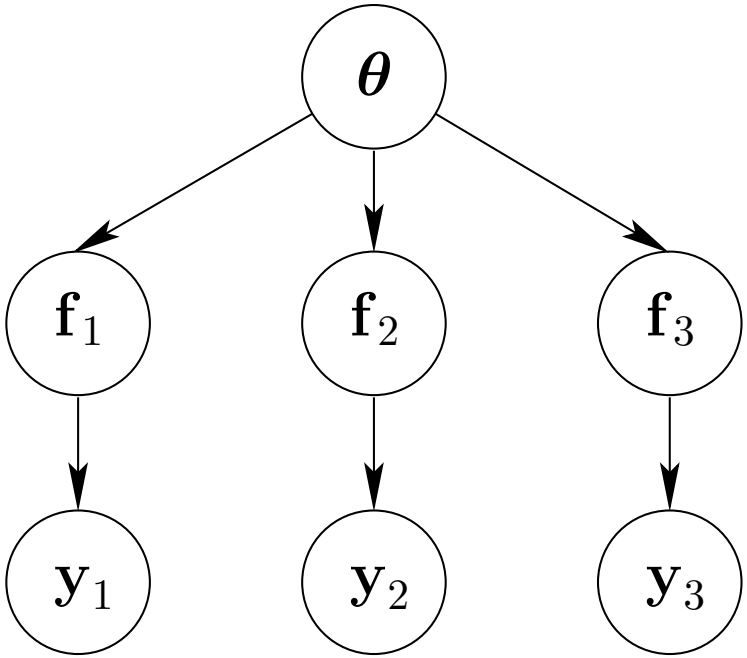

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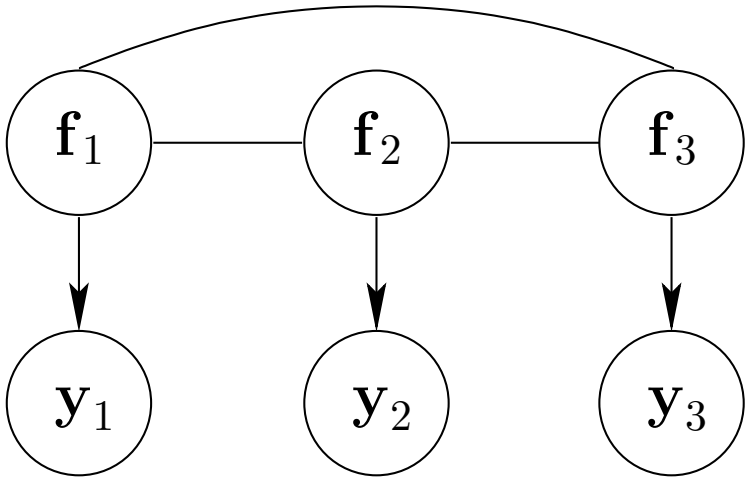

Our approach

- Block diagonal covariance matrix, and each of the M blocks is induced from the same kernel function (Minka and Picard, 1999; Lawrence and Platt, 2004; Yu et al, 2005; Schwaighofer et al, 2005)
- Our model: Observations on one task affect predictions on the others

Multi-task GP

We place a (zero mean) GP prior over the latent functions $\{f_\ell\}$:

The Model

$$\langle f_\ell(\mathbf{x}) f_m(\mathbf{x}') \rangle = K^f_{\ell m} k^x(\mathbf{x}, \mathbf{x}') \qquad y_{i\ell} \sim \mathcal{N}(f_\ell(\mathbf{x}_i), \sigma^2_\ell),$$

K^f: PSD matrix that specifies the inter-task similarities

k^x: Covariance function over inputs

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Correlations between tasks modelled directly via K^f

Multi-task GP Models

K^{f} can be:

- Full non-parametric: General PSD matrix, e.g. $\mathrm{K}^{\mathrm{f}} = (\mathrm{L}^{\mathrm{f}})(\mathrm{L}^{\mathrm{f}})^{\mathrm{T}}$

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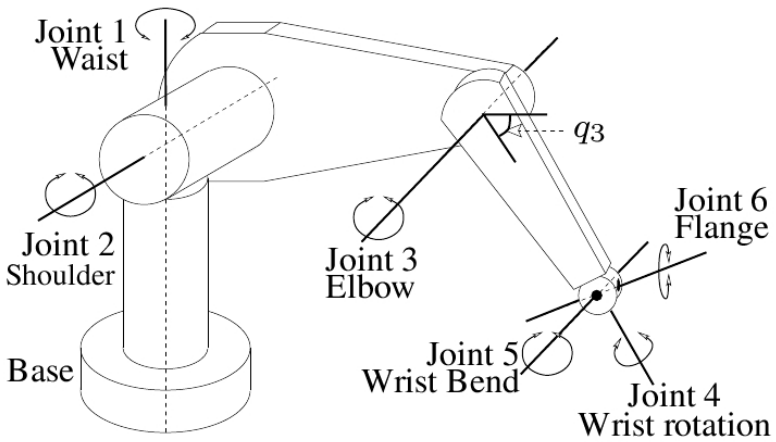
- Block diagonal: Implements task clustering. Cluster structure can be specified a priori. e.g. K^{f} is diagonal (all tasks are independent)
- Mixture: All functions are independent except for one, which is a mixed version of the others. Effective for transferring to a new task:

$$\mathrm{K}^{\mathrm{f}} = \begin{pmatrix} \mathrm{I} & \boldsymbol{\pi} \\ \boldsymbol{\pi}^{\mathsf{T}} & \boldsymbol{\pi}^{\mathsf{T}}\boldsymbol{\pi} \end{pmatrix},$$

where $\boldsymbol{\pi}$ are mixing proportions, and may depend on task descriptors.

Learning Robot Inverse Dynamics (Chai et al, NIPS 2009)



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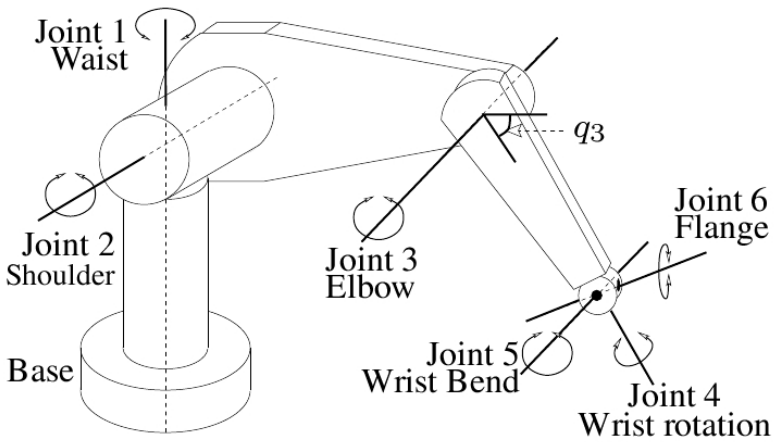

- $\boldsymbol{\tau}$: Torques needed at joints to drive a trajectory $\mathbf{x} = (\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$
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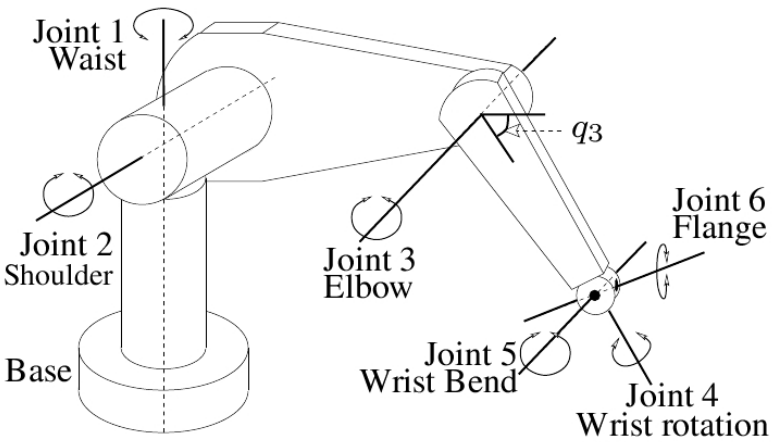

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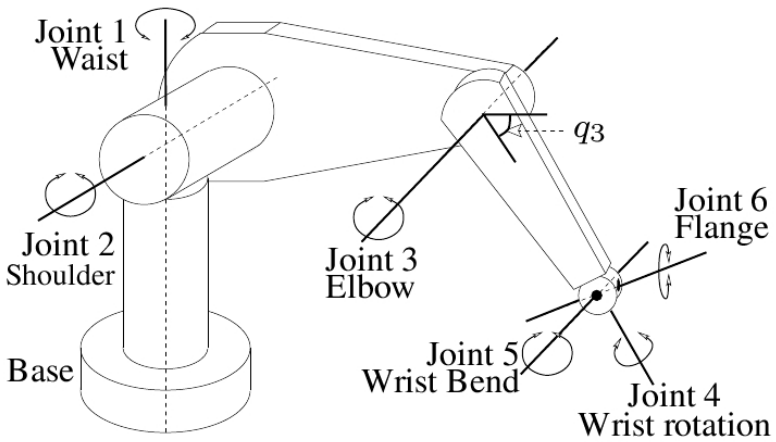

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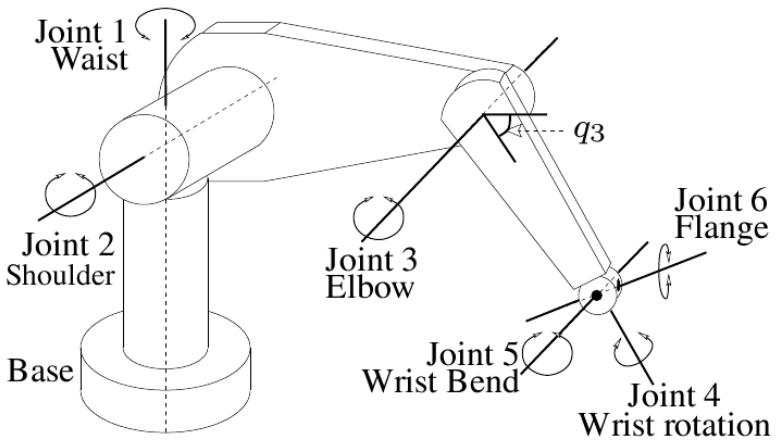

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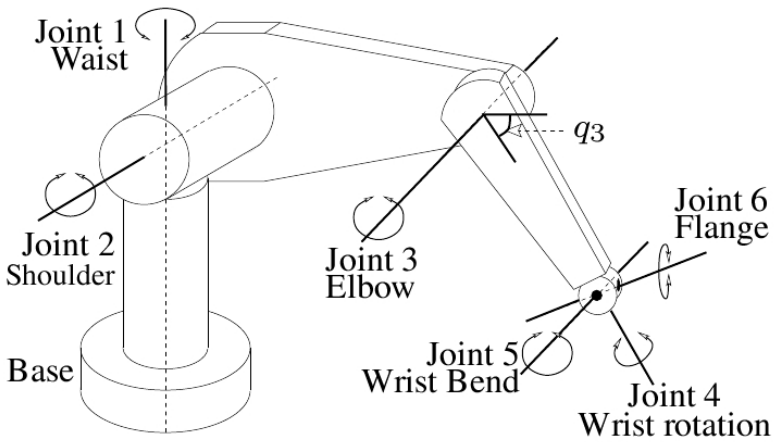

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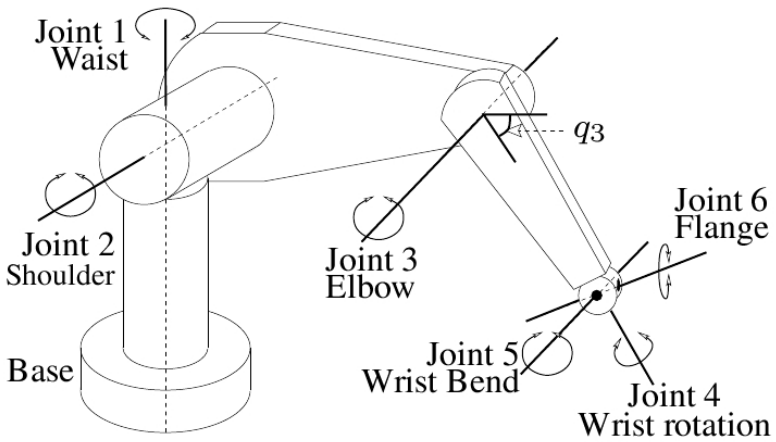

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The MTGP model matches the correlations between torque functions

Other Non-Gaussian Likelihood Models

- We have encountered this in GP classification
- Ordinal regression: Chu and Ghahramani, JMLR 2005
- Preference Learning: Chu and Ghahramani, ICML 2005
- Preference Elicitation (PE): Bonilla et al, NIPS 2010 (to appear)
  - Make optimal recommendations to users by actively querying their preferences.
  - Bayesian decision-theoretic PE approach
  - Correlated GP prior over user's latent utility functions
  - Reduce elicitation burden by leveraging information from previous users

Latent Variable Models

The Gaussian Process Latent Variable Model (GPLVM; Lawrence, NIPS 2004) is a probabilistic model for non-linear dimensionality reduction.

- Main idea: Some high-dimensional data can be embedded into a low-dimensional non-linear manifold.
- model each dimension of $\{\mathbf{x_i}\}_{i=1}^{N}$ with a corresponding latent point $\mathbf{z_i}$ through a non-linear mapping.
- Use an independent GP for this mapping
- Likelihood maximization in order to find the latent projection $\mathbf{z_i}$
- GP models for pose estimation: `http://grail.cs.washington.edu/projects/styleik`

Modeling of Human Poses with GPLVM

Grochow et al, SIGGRAPH 2004

- Style-Based Inverse Kinematics: Given a set of constraints, produce the most likely pose.
- Feature vectors are derived from pose information (e.g from mo-cap data).
  - joint angles, vertical orientation, velocity and accelerations.
- The problem is inherently underdetermined but some poses are more likely than others.
- Low dimensional representations are learned from previous poses using GPLVM
- GPLVM predictive distribution is used in objective function to find new poses given the constraints.

Pose Estimation Movies

From `http://grail.cs.washington.edu/projects/styleik`

Style Pitch

Style Track

Pose Track

Image Pose Basketball

Image Pose Baseball

Interpolation

Conclusions and Future Directions

- GPs as flexible non-parameteric Bayesian technique for regression, classification and other machine learning problems.
- The covariance function is a crucial component in GPs.
- Analytic solutions for standard regression setting and approximate inference for classification.
- Computational issues dealt with through the idea of inducing variables

- More work on design of covariance functions needed
- Towards real large scale GPs
- Dealing with non-standard settings, e.g. preference learning and multi-task learning
- Dealing with structured data

GP Quiz