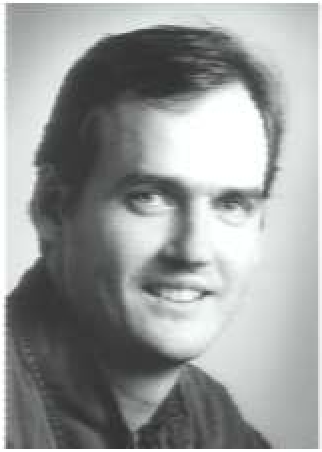
How to Hedge an Option Against an Adversary: Black-Scholes Pricing is Minimax Optimal

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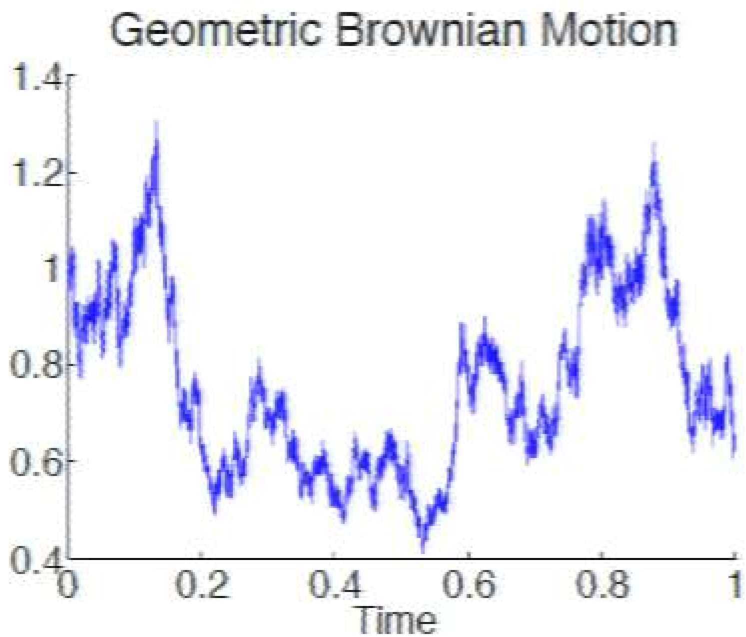
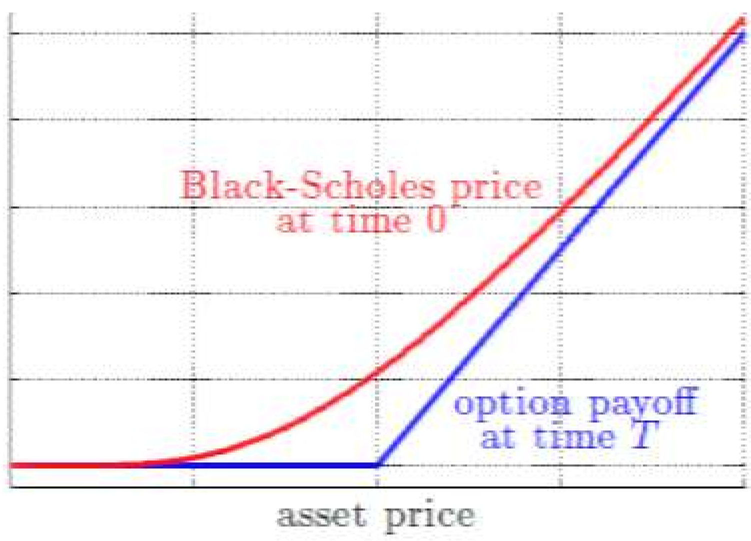
NIPS 2013 Spotlight

Black-Scholes Option Pricing

- Given a financial contract with known payoff at time T, how much is it worth now?
- **Black-Scholes (1973)**: model asset price as continuous-time random walk in log space
- Dynamic hedging strategy $\rightarrow$ replicate option payoff $\rightarrow$ fair price

$$\text{price at time } 0 = \mathbb{E}\left[\,\text{payoff}\,(\,\text{asset price at time } T\,)\right]$$





Robust Option Pricing via Regret Minimization

- Hedging strategy $\equiv$ online learning algorithm
- Can we construct a trading strategy that is robust to **adversarially chosen** price?

Investor
- Observes asset price S
- Invests $\$\Delta$

Market
- Selects fluctuation r
- Updates price $S \leftarrow S(1+r)$

Investor profits Δr

- Investor's goal is to minimize his **regret**:

$$\underbrace{g\Big(S \cdot \prod_{i=1}^{n}(1+r_i)\Big)}_{\text{payoff of option}} - \underbrace{\sum_{i=1}^{n} \Delta_i r_i}_{\text{profit from trading}}$$

- Optimal regret is equivalent to "**minimax value of option**"

Black-Scholes Price is Minimax Optimal

- Analyze minimax regret:

$$V_\zeta^n(S, c) = \inf_{\Delta_1} \sup_{r_1} \cdots \inf_{\Delta_n} \sup_{r_n} \left\{ g\Big(S \cdot \prod_{i=1}^n (1 + r_i)\Big) - \sum_{i=1}^n \Delta_i r_i \right\}$$

with **cumulative volatility** and **maximum jump** constraints:

$$\sum_{i=1}^n r_i^2 \le c, \qquad |r_i| \le \zeta$$

Theorem: If payoff g is convex and Lipschitz, and $\zeta_n \to 0$, then

$$\lim_{n\to\infty} V_\zeta^n(S, c) = U(S, c) \quad \text{[Black-Scholes price]}$$

- Explicit trading strategy gives finite-horizon upper bound on regret:

$$V_\zeta^n(S; c) \le U(S, c) + O(c\,\zeta^{1/4})$$