Matrix Factorization with Binary Components

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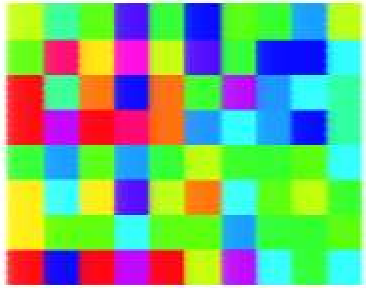
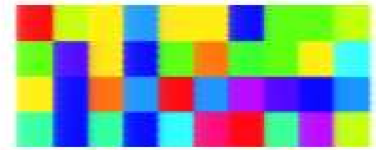
Poster 84

Problem: low-rank matrix factorization with one binary factor

$$D \approx T \quad A$$

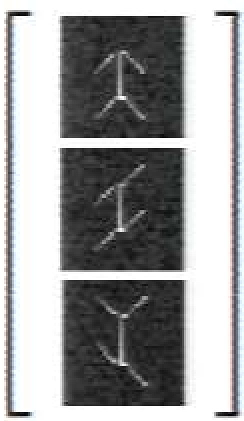


$$\approx \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

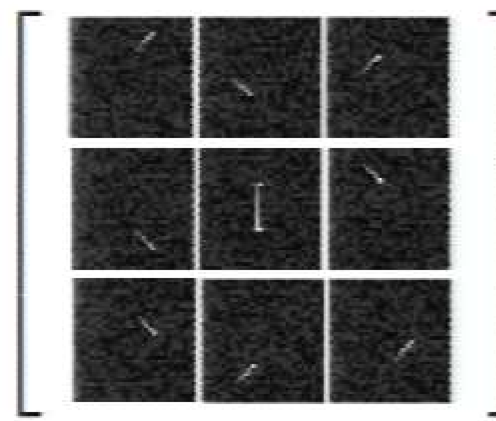


$\mathbb{R}^{m \times n}$ $\quad \{0,1\}^{m \times r}$ $\quad \mathbb{R}^{r \times n}, A^\top \mathbf{1}_r = \mathbf{1}_n$

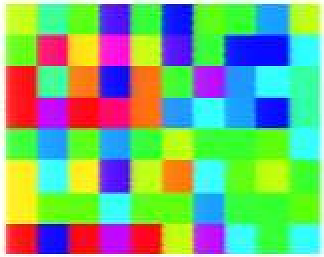
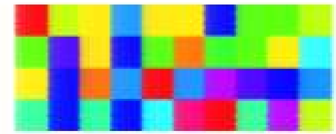
1st **interpretation**: linear combination of binary components
2nd **interpretation**: addition of parts



$$= \left[\begin{matrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix} \right]$$



Exact case:

(P)  $= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ 

$m \times n$ $\quad$ $m \times r$ $\quad$ $r \times n$

Challenges:

- Constrained low-rank matrix factorization (NMF, ...): in general hard to find globally optimal solution.
- Combinatorial constraints. **Naive**: $2^{m \cdot r}$ possibilities !

Previous approaches to problem (P) lack guarantees.

Main Contribution

Provably correct algorithm w/runtime $O(mr2^r) + O(mnr)$.

In practice, we can solve problems up to $r = 80$.

Uniqueness of the factorization is crucial for **interpretability**.

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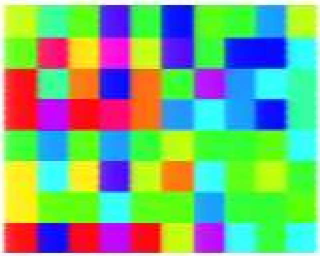
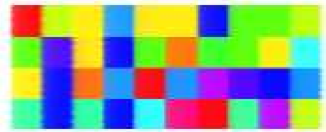


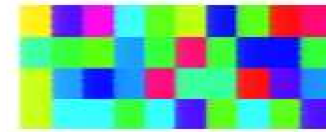
$$= \; T \; \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

A



$$\stackrel{?}{=} \; T' \; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

A'



Results on uniqueness

- Factorization is **unique** if T is **separable**.
- Factorization is **unique** with high probability as $m \to \infty$ for binary matrices T generated uniformly at **random**.

Implication for our algorithm:
Conditions on $T \implies$ can solve variants of the problem w/additional constraints (e.g. non-negativity) on A.