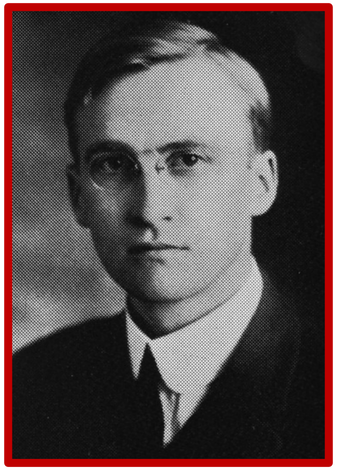
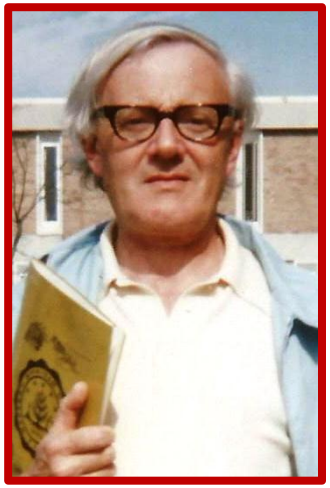
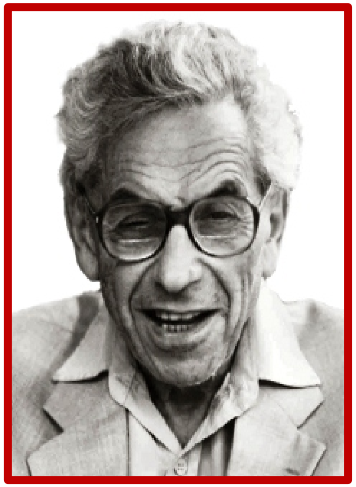
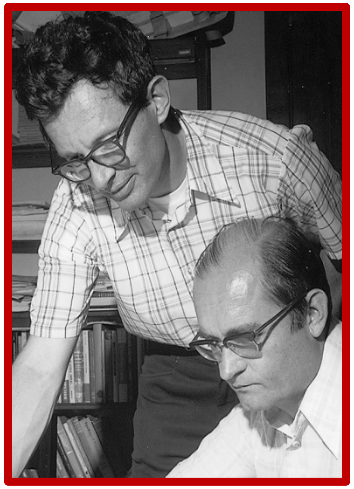
A Century of Graph Theory

Robin Wilson







Graph theory: 1840–1890

1852: The 4-colour problem is posed
1879: Kempe 'proves' the 4-colour theorem
1880: Tait introduces edge-colourings

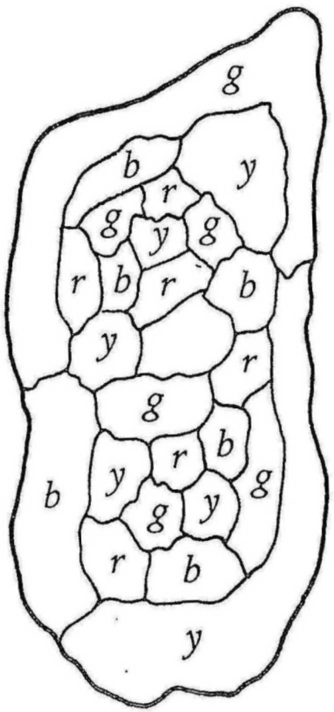
1855–57: Kirkman and Hamilton on cycles
1871: Hierholzer on Eulerian graphs

1845: Kirchhoff introduces spanning trees
1857–75: Cayley counts trees and molecules
1878: Sylvester's chemistry and 'graphs'
1889: Cayley's n^{n-2} theorem

1861: Listing's topological complexes

1890: Percy Heawood

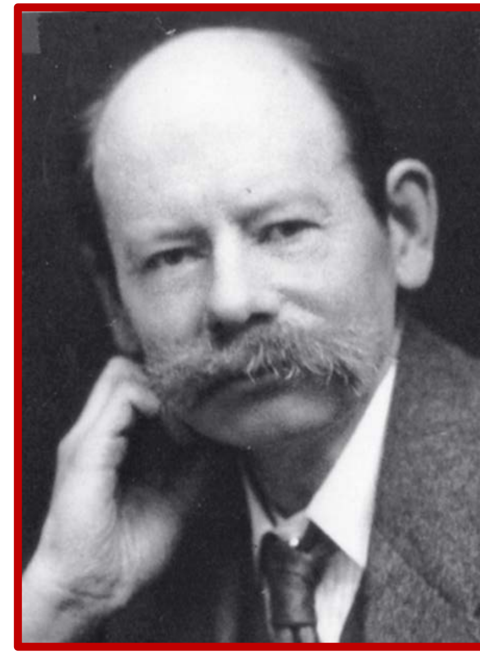
Map-colour theorem

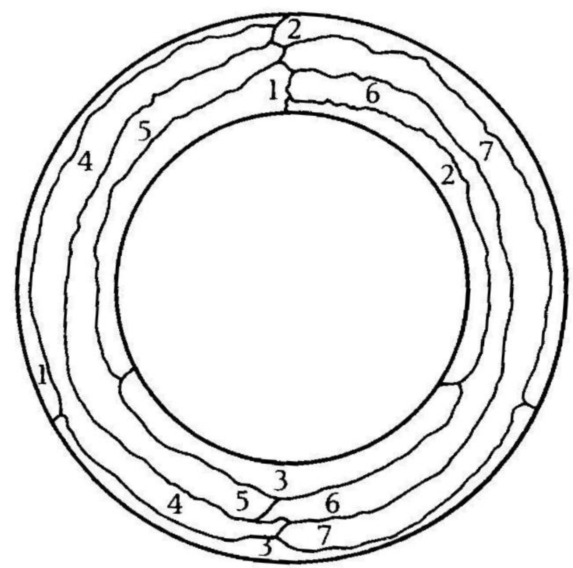


Heawood
pointed out the error
in Kempe's 'proof'

salvaged enough to prove
the five-colour theorem

for maps on a surface
of genus g (≥ 1)
$[(7 + \sqrt{(1 + 48g)})/2]$
colours are sufficient





1891: Lothar Heffter

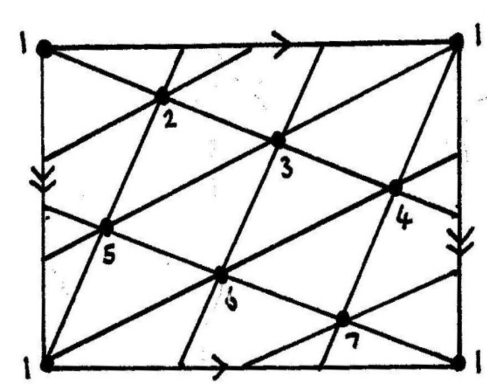
Ueber das problem der Nachbargebiete

For $g > 1$, Heawood didn't prove that
$[(7 + \sqrt{(1 + 48g)})/2]$ colours are actually needed

Heffter noticed the omission and asked (equivalently):
What's the least genus for n neighbouring regions on the surface? For $n \geq 7$ it's at least $\{(n – 3)(n – 4)/12\}$

Heffter proved this for $n \leq 12$ and some other values

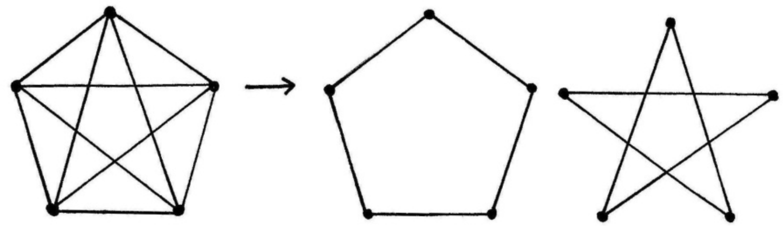
He also 'dualized' the problem to embedding complete graphs on a surface: what's the least genus g for the graph K_n?



1891/1898: Julius Petersen

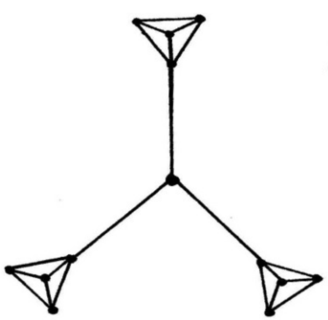
Die Theorie der regulären Graphs

When can you factorize a regular graph into regular 'factors' of given degree r?

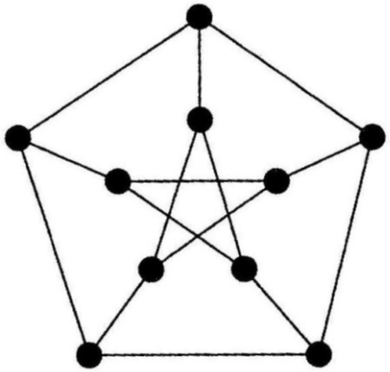


K_5 has a '2-factorization', as does every regular graph of even degree

Sylvester: this graph has no 1-factor



The Petersen graph splits into a 2-factor and a 1-factor, but not three 1-factors



1892: W. W. Rouse Ball

Mathematical Recreations and Problems

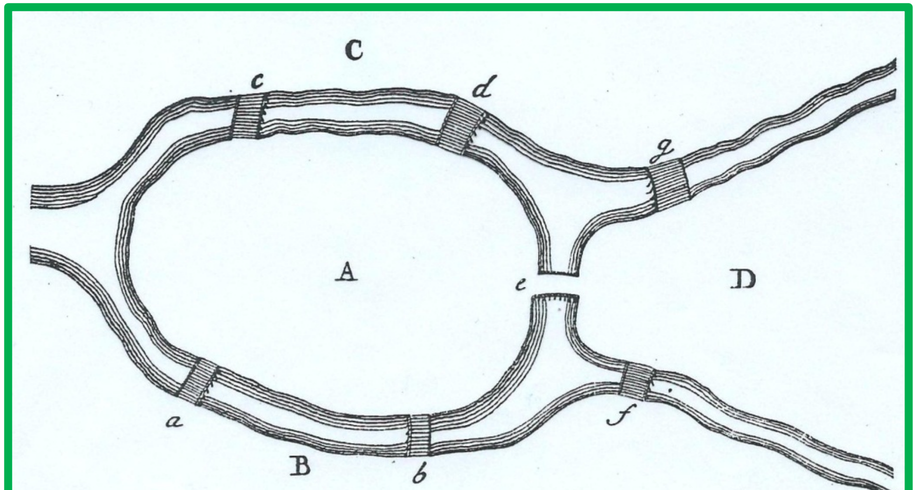
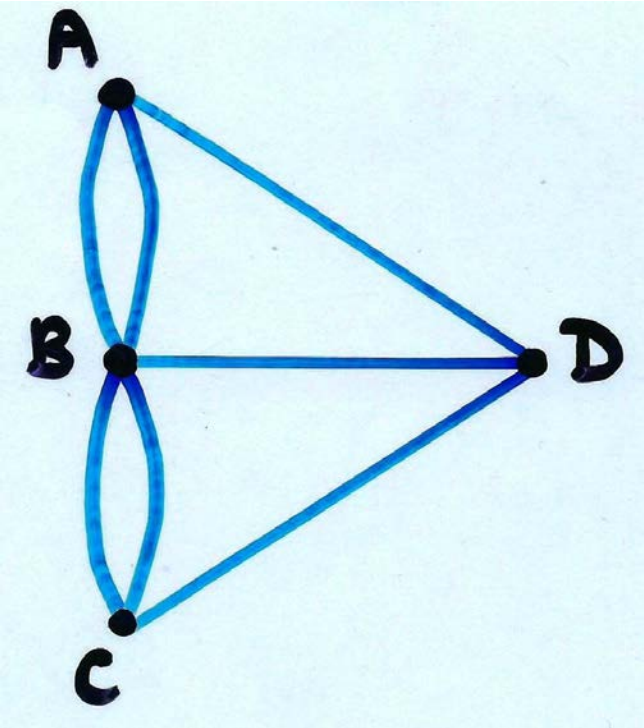


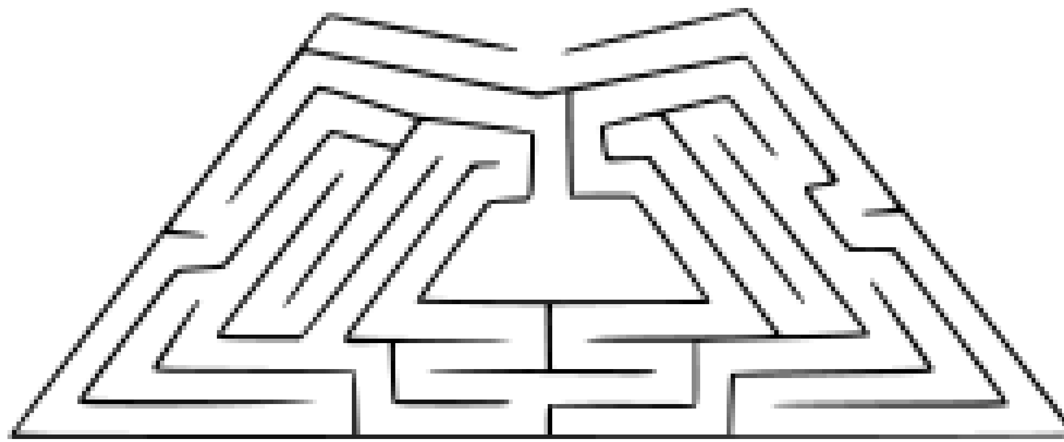
Fig. 1.



Solving the Königsberg
bridges problem corresponds
to drawing the right-hand picture
without repeating any line
or lifting your pen from the paper

Euler did NOT draw
such a picture

1895: Gaston Tarry
The problème des labyrinthes



Tarry's rule: do not return along the passage which has led to a junction for the first time unless you cannot do otherwise.

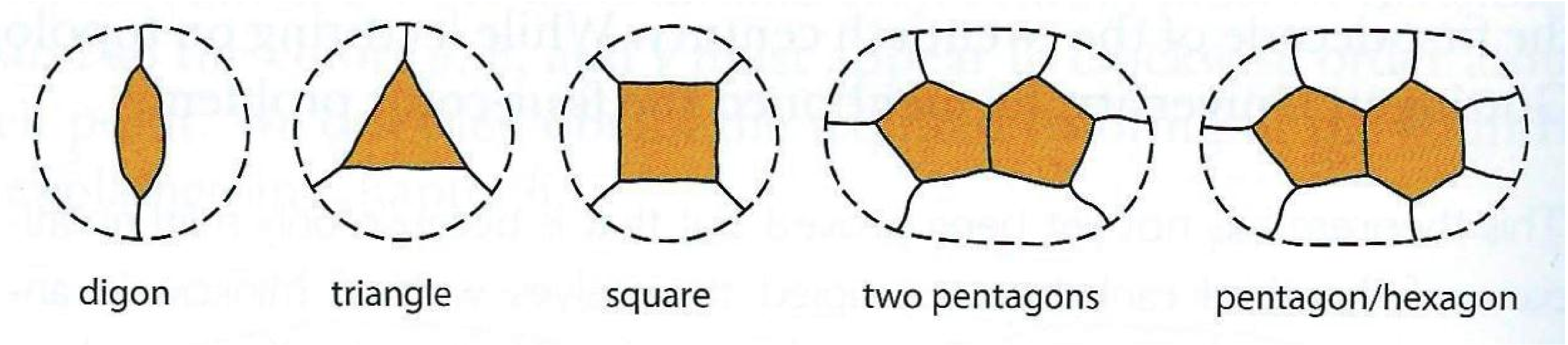
He also gave a practical method for carrying this out.

1904: Paul Wernicke

Über den kartographischen Vierfarbensatz

Kempe: Every map on the plane contains a digon, triangle, square or pentagon

Wernicke: Every map on the plane contains at least one of the following configurations



They form an 'unavoidable set':
every map must contain at least one of them

1907: M. Dehn & P. Heegaard

Analysis situs

Encyklopädie der Mathematische Wissenschaften

First comprehensive study of complexes,
following on from ideas of Kirchhoff,
Listing and Poincaré

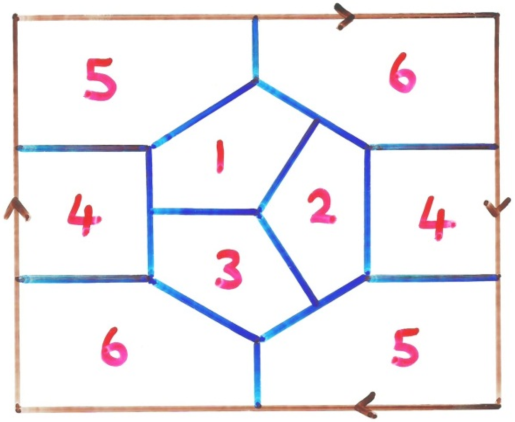
Their opening section was on Liniensysteme (graphs)
constructed from 0-cells (vertices) and 1-cells (edges)

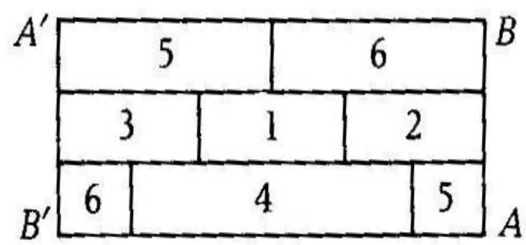
This work was later continued by Oswald Veblen
in a paper on Linear graphs (1912)
and in an American Mathematical Society
Colloquium Lecture series in 1916

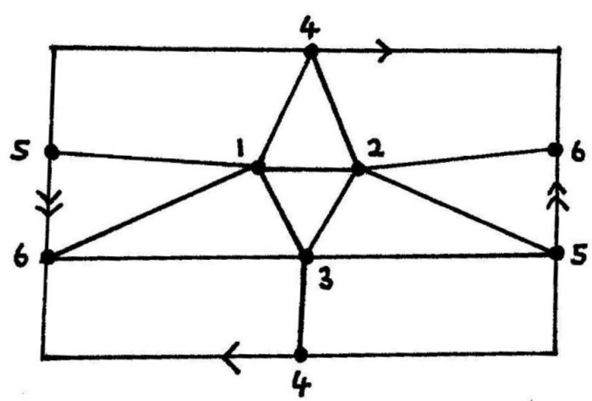
1910: Heinrich Tietze

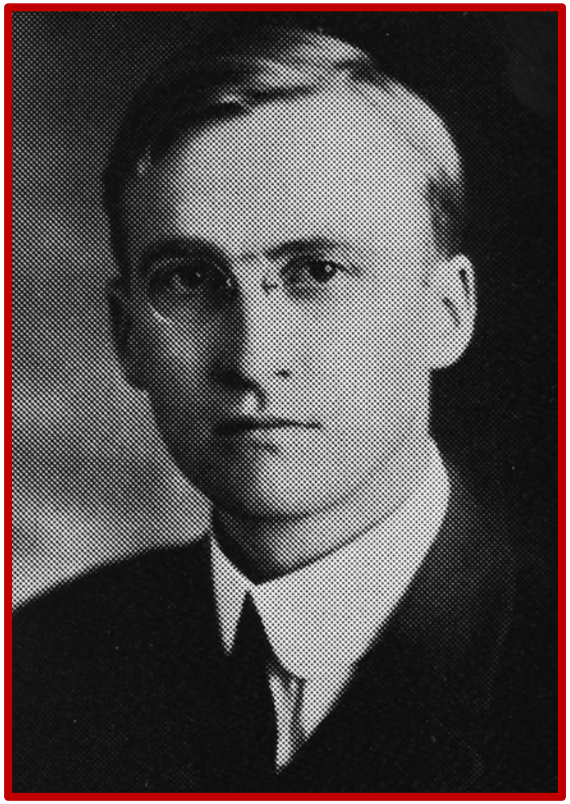
Eine Bemerkungen über das Problem des Kartenfärbens auf einseitigen Fläschen

One-sided surfaces: on a Möbius band or projective plane,
every map can be coloured with 6 colours
so at most 6 neighbouring regions can be drawn
Klein bottle: 7 colours are needed (Franklin, 1934)
Tietze also obtained analogues of the formulas
of Heawood and Heffter









1912: G. D. Birkhoff

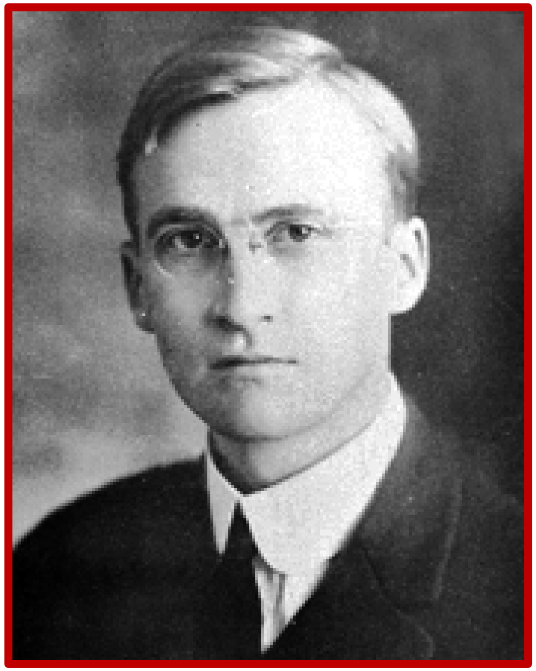
A determinant formula for the number of ways of coloring a map

The number of ways is always a polynomial in the number of colours, now called the **chromatic polynomial**

The degree is the number of countries and the coefficients alternate in sign: Birkhoff obtained a formula for them

Related work by Birkhoff (1930), Whitney (1932), and in a major paper by Birkhoff and D. C. Lewis (1944)

1913: G. D. Birkhoff
The reducibility of maps

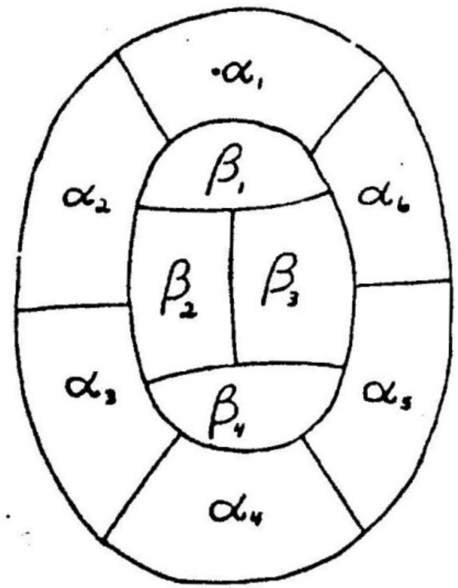


A configuration of countries in a map is reducible if any 4-colouring of the rest of the map can be extended to the configuration.

Irreducible configurations cannot appear in counter-examples to the 4-colour theorem

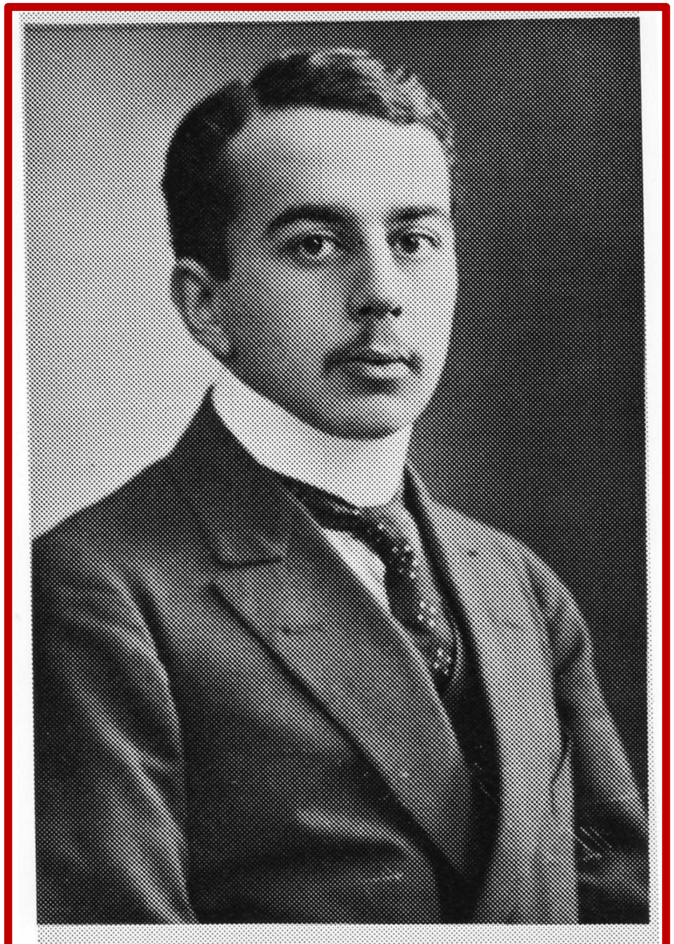
Kempe: digons, triangles and squares are reducible

Birkhoff: so is the Birkhoff diamond



1916: Dénes König

Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre



A graph is bipartite ↔ every cycle has even length

Every regular bipartite graph of degree k splits into k 1-factors
(Interpretation for matching/marriage)

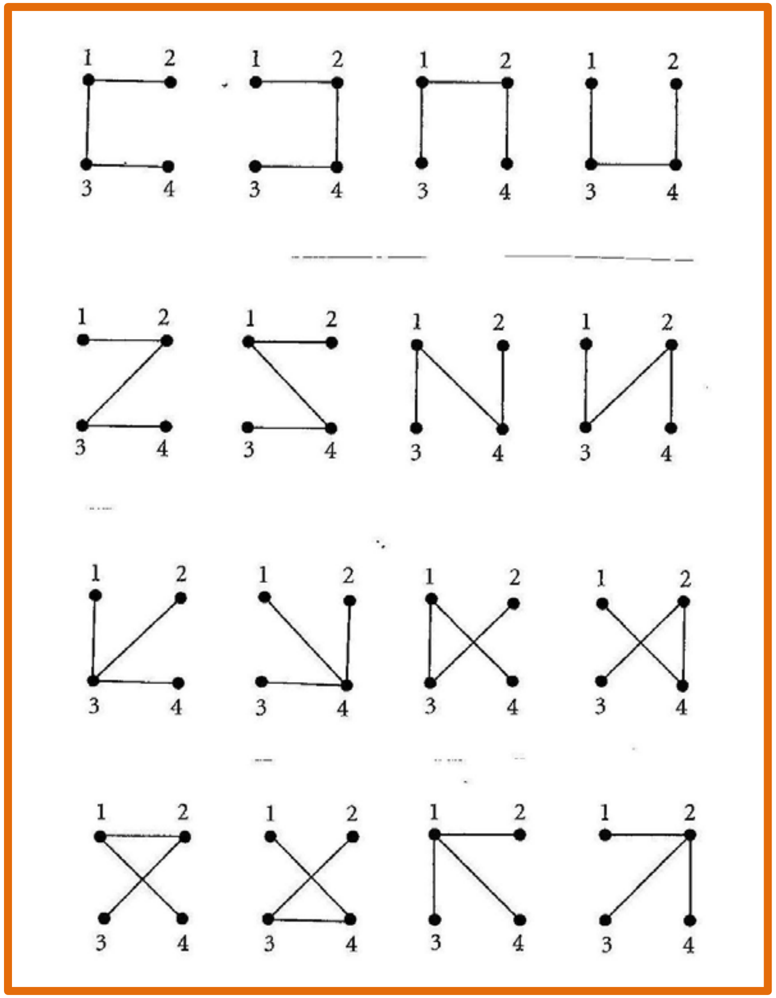
So if each vertex of a bipartite graph G has degree ≤ k,
then the edges of G
can be coloured with k colours

1918: Heinz Prüfer

Neuer Beweis eines Satzes über Permutationen

First correct proof
of Cayley's 1889 result:

There are n^{n-2} labelled trees
on n vertices
or
K_n has n^{n-2} spanning trees

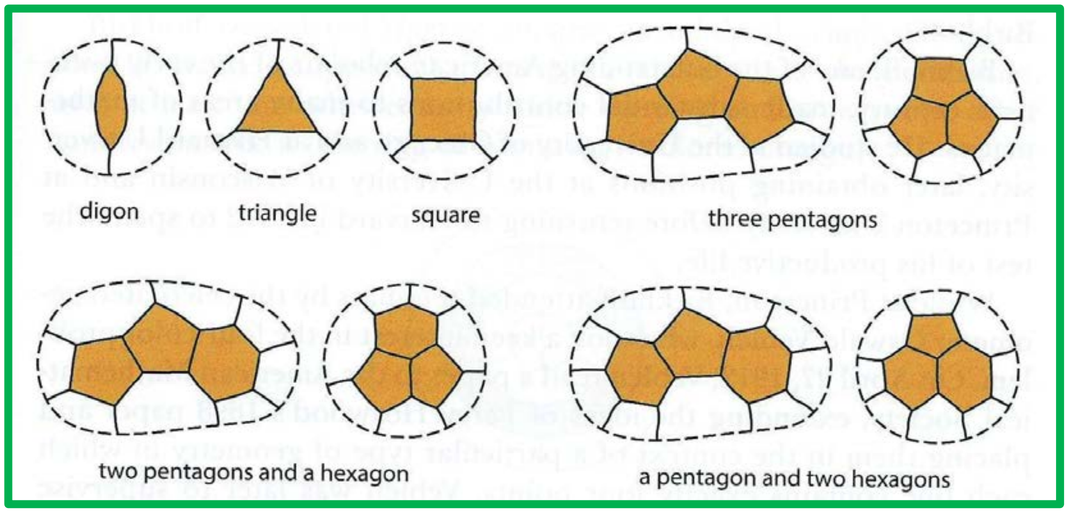


It uses the idea of a Prüfer sequence for each tree

1922: Philip Franklin

The four color problem

Every cubic map containing no triangles or squares must have at least 12 pentagons



Any counter-example has at least 25 countries

later extended by Reynolds (27), Winn (39) and others

Further unavoidable sets found by H. Lebesgue (1940)

1927: Karl Menger

Zur allgemeinen Kurventheorie

On a problem in analytic topology:

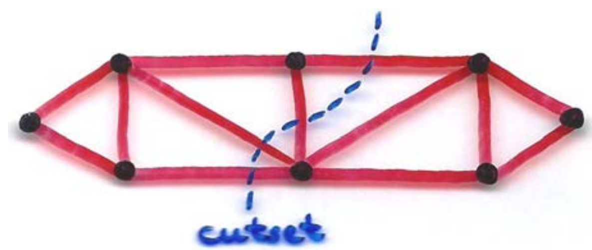
in graph theory terms

it's a **minimax** connectivity theorem

on the number of disjoint paths between

two vertices and how many vertices/edges

must be removed to separate the graph



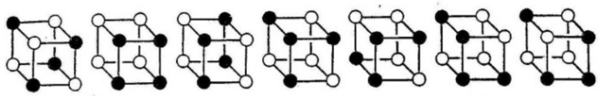
— equivalent to König's theorem (1916)

and Hall's 'marriage' theorem (1935)

1927: J. Howard Redfield

The theory of group-reduced distributions

The actual configurations, shown below, cannot be determined by the methods of the present theory, but must be found, as in all other cases, by detailed consideration of the groups involved, and this may of course be very laborious, except in simple cases, or where special devices are available.



(1) (2) (3) (4) (5) (6) (7)

In connection with the present example we may note without proof certain other simple results obtainable.

Thus if in V we substitute $x^r + y^r$ for every s_r, we obtain the polynomial

$$x^8 + x^7y + 3x^6y^2 + 3x^5y^3 + 7x^4y^4 + 3x^3y^5 + 3x^2y^6 + xy^7 + y^8,$$

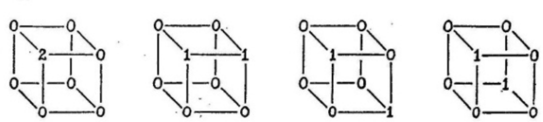
in which the coefficient of $x^t y^{8-t}$ enumerates the distinct configurations possible with t nodes • and $8 - t$ nodes ∘.

The sum of the coefficients in the above expression is 23, which is the total number of configurations when the numbers of nodes of the two colors are not specified. This enumeration is also effected by substituting 2 for every s_r in V. Similarly if k colors are available we substitute k for every s_r; thus with 3 colors there are $(1/24)(3^8 + 9.3^4 + 8.3^4 + 6.3^2) = 333$ possible configurations.

If in V we put $1/(1 - x^r)$ for every s_r, we obtain the infinite series

$$1 + x + 4x^2 + 7x^3 + 21x^4 + 37x^5 + \cdots,$$

in which the coefficient of x^t enumerates the distinct configurations obtained by placing a zero or a positive integer at every vertex of the cube, subject to the condition that the sum of the 8 numbers is always t. For $t = 2$, the 4 configurations are



If in V we put 2 for every s_{2k} and 0 for every s_{2k+1}, we enumerate the configurations in which it is possible to change the color of every node into

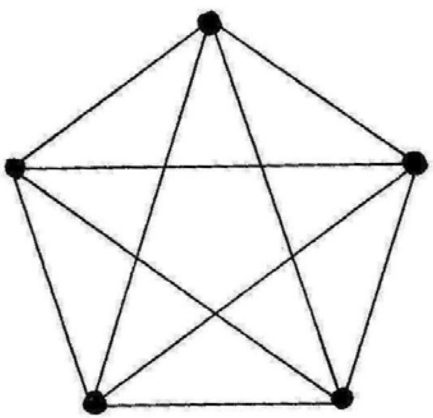
Counting under symmetry, counting simple graphs

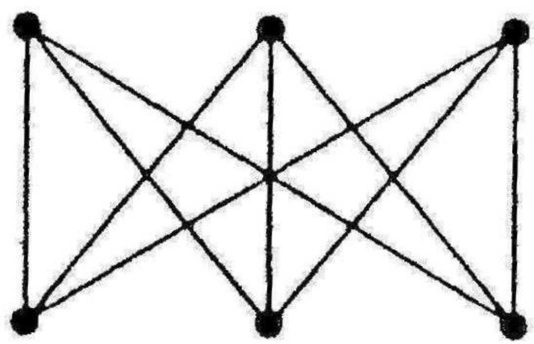
(symmetrical aliorative dyadic relation-numbers)

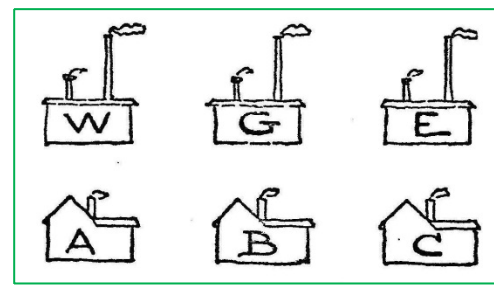
1930: Kasimierz Kuratowski

Sur le problème des courbes gauches en topologie

A graph is planar if and only if it doesn't contain K_5 or $K_{3,3}$







The utilities puzzle of Sam Loyd

Proved independently by O. Frink and P. A. Smith

1930: F. P. Ramsey

On a problem in formal logic

Ramsey graph theory

Example: Six people at a party

Every colouring of the edges of K_6
in two colours red and blue
gives either a red triangle
or a blue triangle

With k colours, how many vertices do we need
to guarantee a given graph of one colour?

1931–35: Hassler Whitney

1931: Non-separable and planar graphs
1931: The coloring of graphs

1932: A logical expansion in mathematics
1932: Congruent graphs and the connectivity of graphs

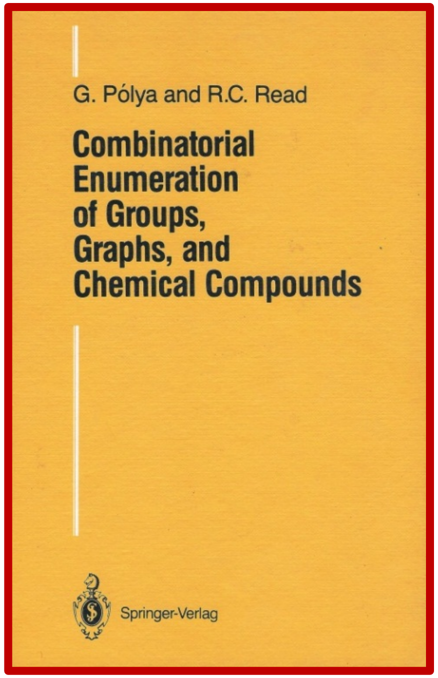
1933: A set of topological invariants for graphs
1933: 2-isomorphic graphs
1933: On the classification of graphs

1935: On the abstract properties of linear dependence

1935–37: Georg Pólya

Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen

On enumerating graphs and chemical molecules, the orbits under a group of symmetries, using the cycle structure of the group

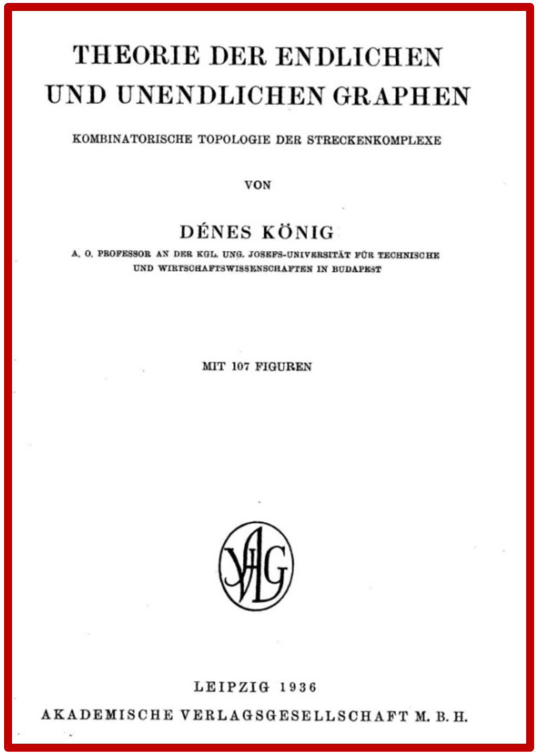


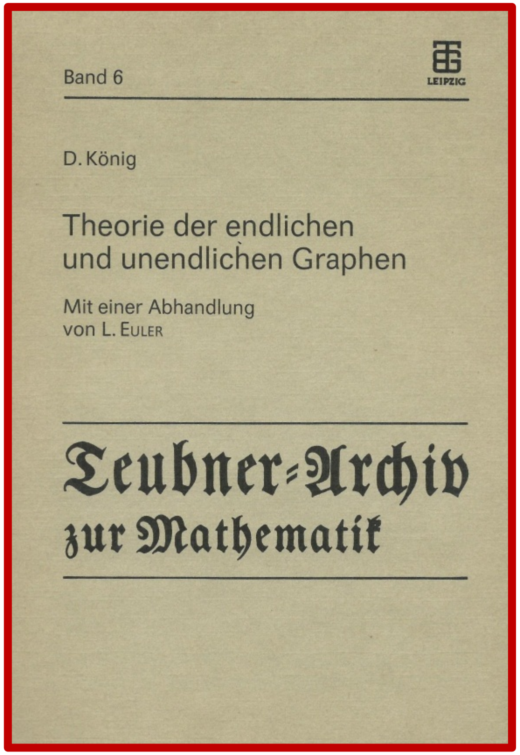
Later work by Otter, de Bruijn, Harary, Read, Robinson, etc.

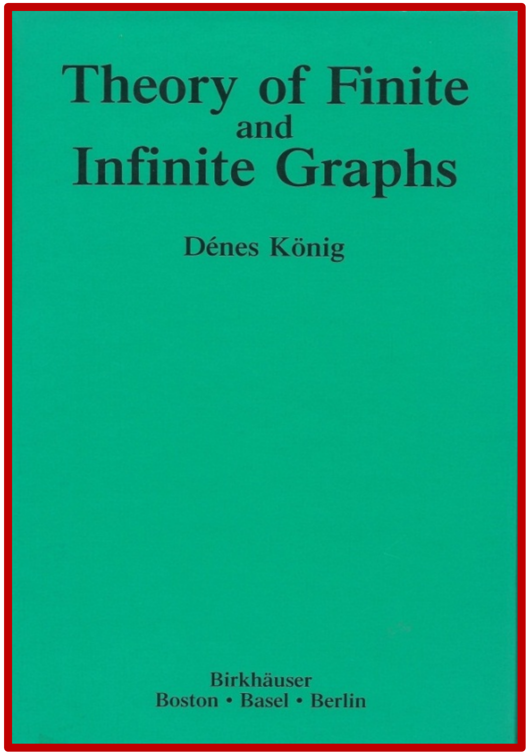
1936: Dénes König

Theorie der endlichen und unendlichen Graphen

The ‘first textbook on graph theory’

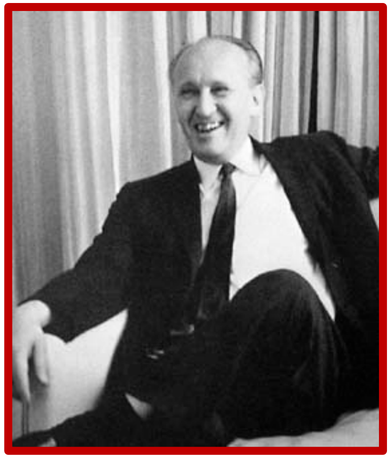






1940: P. Turán

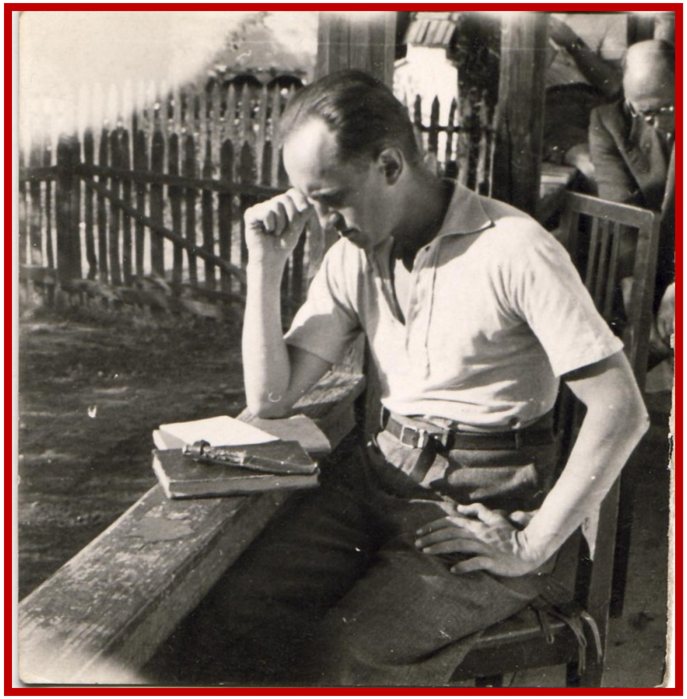
Eine Extremalaufgabe aus der Graphentheorie



Extremal graph theory

A graph with n vertices
and no triangles
has ≤ $[n^2/4]$ edges

[proved earlier by W. Mantel (1907)]



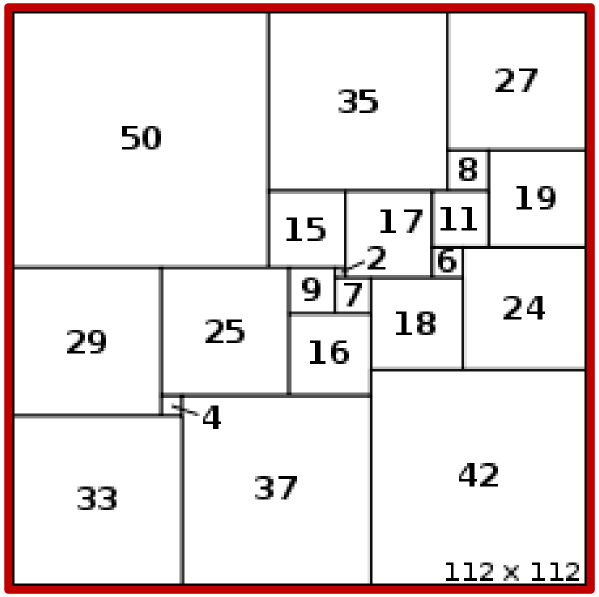
[Turán also studied the 'brick factory problem' on crossing numbers]

1941: R. L. Brooks

On colouring the nodes of a network

If G is a connected graph with maximum degree k, its vertices can be coloured with at most k + 1 colours, with equality for odd complete graphs and odd cycles

[Brooks was one of the team of Brooks, Stone, Smith and Tutte who used directed graphs to 'square the square' in 1940]

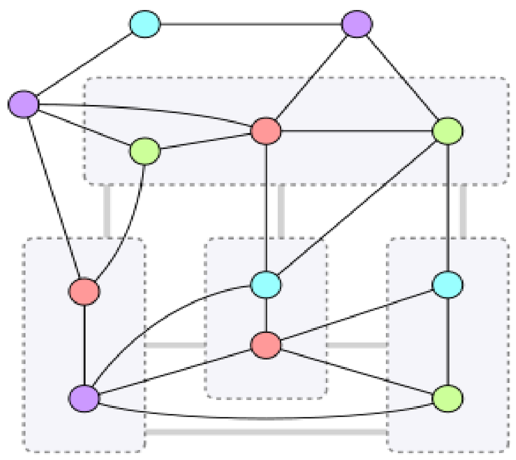


1943: Hugo Hadwiger

Über eine Klassifikation der Streckencomplexe

Hadwiger's conjecture

Every connected graph with chromatic number k can be contracted to K_k



Hadwiger: conjecture true for $k \leq 4$

Wagner (1937): true for $k = 5$ ↔ four-colour theorem

Robertson, Seymour and Thomas (1993): true for $k = 6$

(also uses four-colour theorem)

Still unproved in general

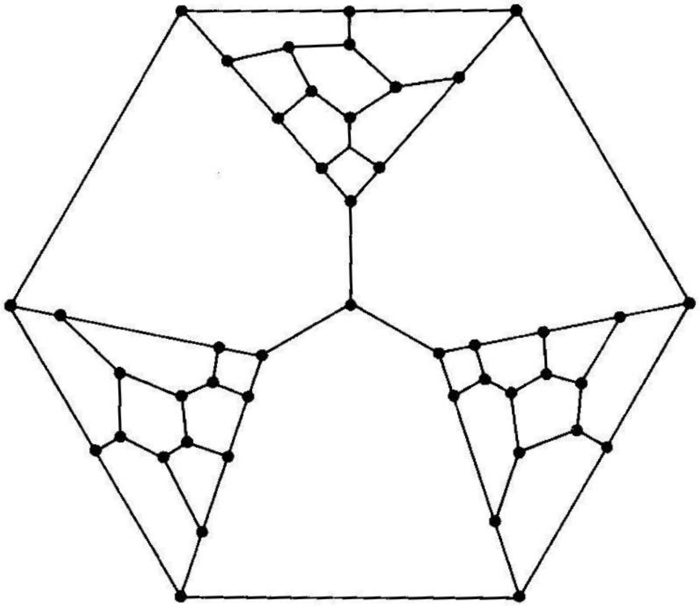
1946: W. T. Tutte

On Hamilton circuits

Tait's conjecture (1880):

Every cubic polyhedral graph has a Hamiltonian cycle

Tutte's graph was the first counter-example: it has 46 vertices



[In 1947 Tutte found a condition for a graph to have a 1-factor (extended to r-factors in 1952)]

1952: Gabriel Dirac

Some theorems on abstract graphs

Sufficient conditions for a graph G to be Hamiltonian
(there is a cycle passing through every vertex)

Dirac (1952): If G has n vertices, and if the degree of each vertex is at least n/2, then G is Hamiltonian

Ore (1960): If deg(v) + deg(w) ≥ n for all non-adjacent vertices v and w, then G is Hamiltonian

Further Hamiltonian results by Pósa, Chvátal, etc.

[Dirac also wrote on 'critical graphs']

Algorithms from the 1950s/1960s

Assignment problem

H. Kuhn (1955)

Network flow problems

L. R. Ford & D. R. Fulkerson (1956)

Minimum connector problem

J. B. Kruskal (1956) and R. E. Prim (1957)

(anticipated by O. Boruvka (1928) and V. Jarník (1930))

Shortest path problem

E. W. Dijkstra (1959)

Chinese postman problem

Kwan Mei-Ko (= Meigu Guan) (1962)

1959: P. Erdős & A. Rényi

On random graphs I

Probabilistic graph theory

G(n, m) model (Erdős–Rényi)

Choose at random a graph with n vertices and m edges

How many components does it have?

How big is its largest component?

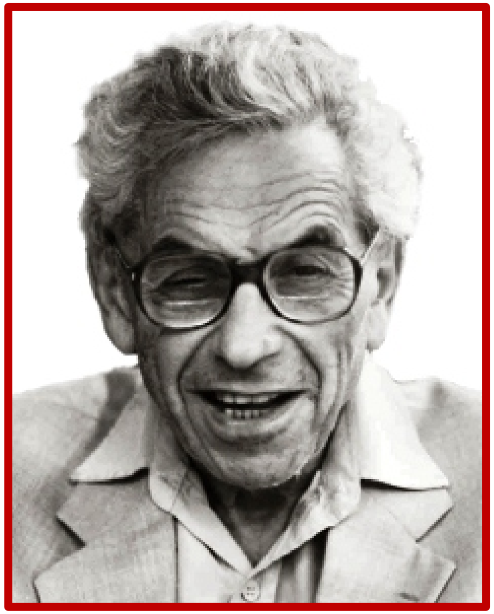
What is the probability that it is connected?

G(n, p) model (E. N. Gilbert)

Take n vertices and add edges at random with probability p

How big is the largest component?

When does the graph become connected?



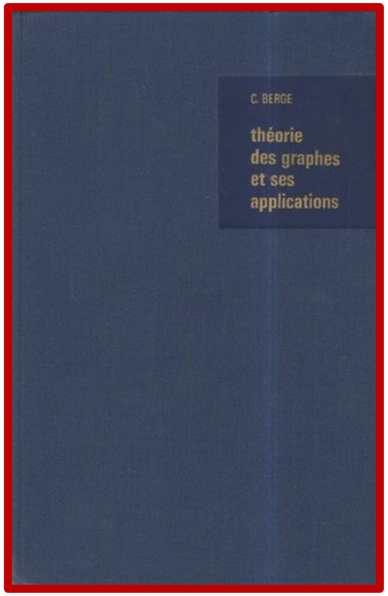
Graph theory textbooks

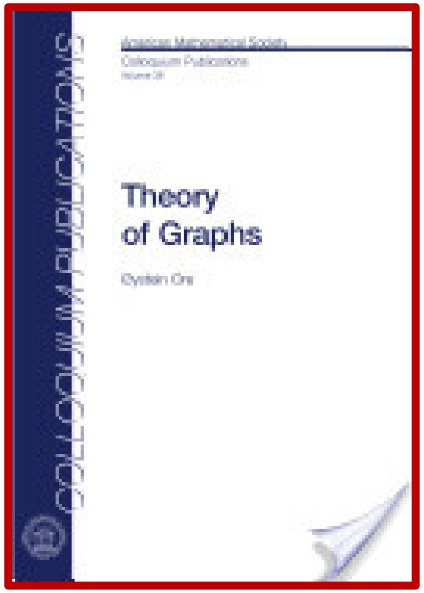
Claude Berge: *Theorie des Graphes et ses Applications* (1958)

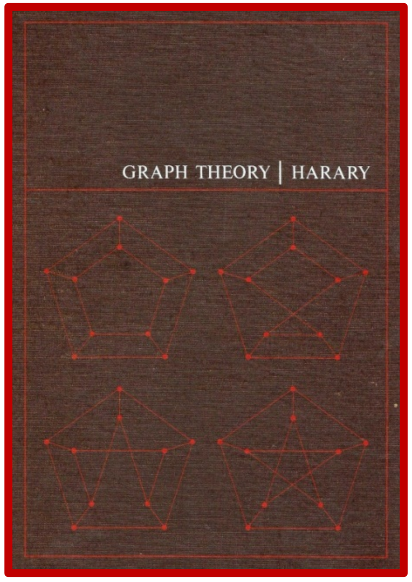
Oystein Ore: *Theory of Graphs* (1962)

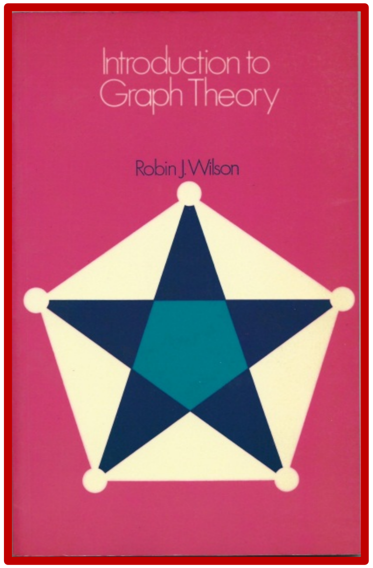
Frank Harary: *Graph Theory* (1969)

Robin Wilson: *Introduction to Graph Theory* (1972)









1964: V. G. Vizing

On an estimate of the chromatic class of a p-graph

The chromatic index $\chi'(G)$ is the smallest number of colours needed for the edges of G so that adjacent edges are coloured differently.

Tait (1880): if G is cubic and planar, then $\chi'(G) = 3$
(4-colour problem)

If Δ is the largest vertex-degree, then:

König (1916): if G is bipartite, $\chi'(G) = \Delta$

Shannon (1949): $\Delta \leq \chi'(G) \leq {}^{3}/_{2}\, \Delta$

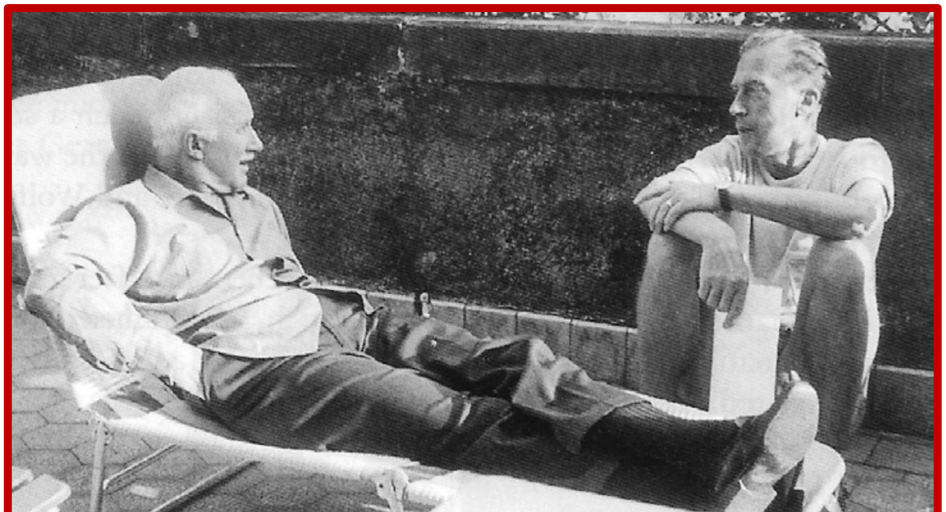
Vizing (1964): if G is simple, $\chi'(G) = \Delta$ or $\Delta + 1$

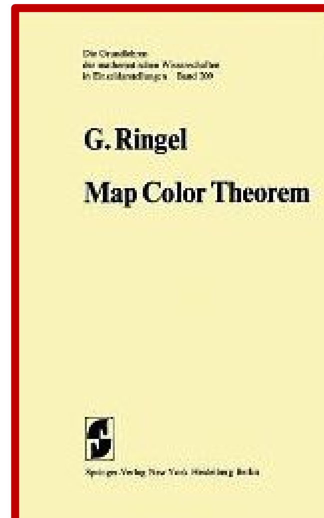
1968: G. Ringel & J. W. T. Youngs

Solution of the Heawood map-coloring problem

Ringel and Youngs reduced the drawing of K_n on a sphere with $\{(n-3)(n-4)/12\}$ handles to twelve cases which they dealt with individually

(The non-orientable case was completed by Ringel in 1952)

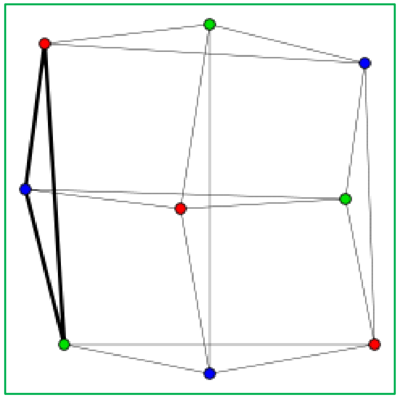




1972: Laszló Lovász

A characterization of perfect graphs

A graph G is perfect if, for each induced subgraph, the chromatic number = the size of the largest clique



Berge graph (1963): neither G nor its complement has an induced odd cycle of length ≥ 5

Lovász (1972): Perfect graph theorem:

A graph is perfect if and only if its complement is perfect

M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas (2006):

Strong perfect graph theorem:

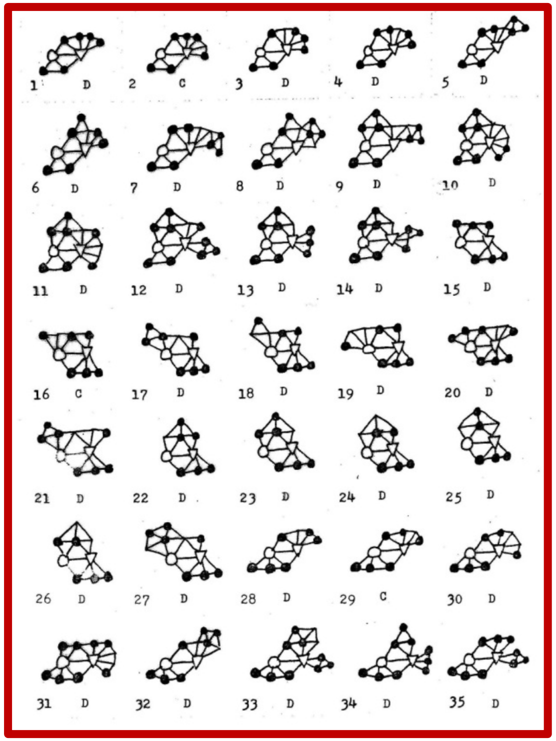
Perfect graphs = Berge graphs

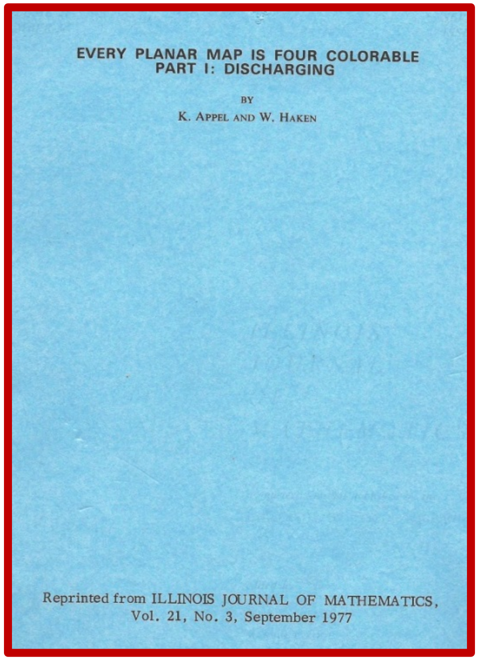
1976: K. Appel & W. Haken

Every planar map is four-colorable

H. Heesch: find an unavoidable set of reducible configurations

Using a computer Appel and Haken (and J. Koch) found an unavoidable set of 1936 reducible configurations (later 1482)

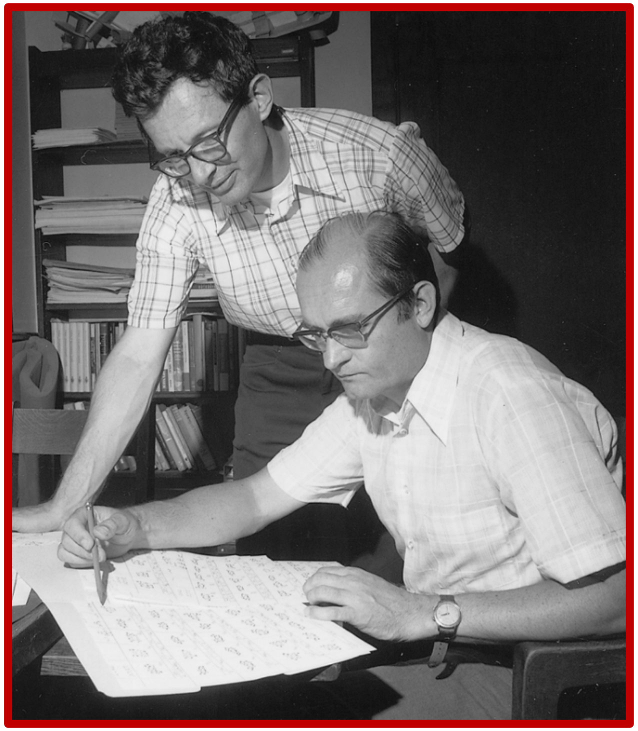



EVERY PLANAR MAP IS FOUR COLORABLE
PART I: DISCHARGING

BY

K. APPEL AND W. HAKEN

Reprinted from ILLINOIS JOURNAL OF MATHEMATICS,
Vol. 21, No. 3, September 1977



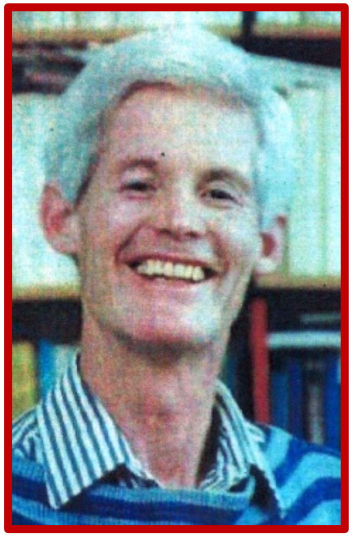
1970s: computational complexity

Efficiency of algorithms

P: 'easy' problems, solved in polynomial time

minimum connector problem (n^2), planarity algorithms (n)

NP: 'non-deterministic polynomial-time problems': given solutions can be checked in polynomial time



Big question: is P = NP?

S. Cook (1971):

The complexity of theorem-proving procedures

***Every* problem in NP can be polynomially reduced to just one problem in NP (the 'satisfiability problem')**

'NP-complete' graph problems:

Hamiltonian cycle problem, graph isomorphism, 3-colourability, travelling salesman problem

1979: H. Glover & J. P. Huneke

The set of irreducible graphs for the projective plane is finite

How many ‘forbidden subgraphs’ are there for a given surface?

Kuratowski (1930): for the sphere, there are two: K_5 and $K_{3,3}$

Conjecture (Erdős, 1930s): for any surface the number is finite

Glover & Huneke (1979) (with D. Archdeacon & C. S. Wang):
for the projective plane the number is 103

For the torus the number is unknown (but is ≥ 800)

Archdeacon & Huneke (1980): it is finite for non-orientable surfaces

Robertson and Seymour (1984): The graph minor theorem
The number is finite for *all* surfaces

1983–2004: N. Robertson & P. Seymour

& co-workers R. Thomas, M. Chudnovsky, . . .

A succession of fundamental results:

- The graph minor theorem
- An improved proof of the 4-colour theorem
- Proof of the Hadwiger conjecture for K_6
- Every snark contains the Petersen graph
- The strong perfect graph conjecture

and many more . . .