Immersions of graphs and digraphs

Bojan Mohar

Simon Fraser University, Burnaby
and IMFM, Ljubljana

(based on joint work with

M. DeVos, Z. Dvořák, J. Fox, J. McDonald, D. Scheide)

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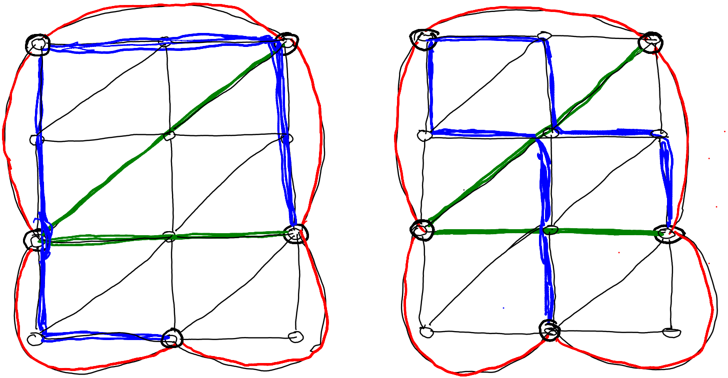
Immersion and strong immersion

- $f : V(H) \to V(G)$ injective
- $\forall uv \in E(H)$ path P_{uv} connecting $f(u)$ with $f(v)$
- Subdivision:
  P_{uv} internally vertex-disjoint
- Immersion:
  P_{uv} edge-disjoint
- Strong immersion:
  P_{uv} edge-disjoint and internally disjoint from $f(V(H))$

Immersion and strong immersion of K_5^-



K_5-immersions in a planar graph

Motivation: Hadwiger and Hajós Conjecture

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Abu-Khzam & Langston: $\chi(G) \le$ largest clique immersion in G

Well-Quasi-Ordering

Robertson and Seymour, Graph Minors XXIII. Finite graphs are well-quasi-ordered for the weak immersion relation.

- The proof uses the full graph minors machinery.
- However, topological obstructions only when cubic vertices ...

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- Polynomial-time algorithms for H-immersion, $O(n^{h+3})$, or membership testing for an immersion-closed family.
- Some indication about WQO for strong immersions.

Grid Theorem

Chudnovsky, Dvořák, Klimošová, Seymour (2014):

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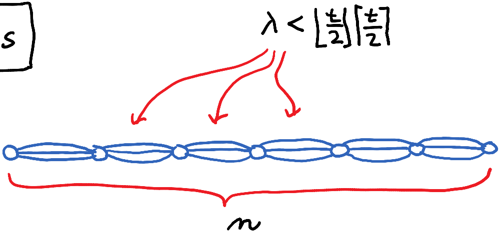
For every fixed k, every 4-edge-connected graph of sufficiently large tree-width contains a $k \times k$ grid as a strong immersion.

$\Rightarrow$ contains any fixed graph H of maximum degree ≤ 4 as a strong immersion

(take a drawing of H inside a large grid)

Two examples

$K_t \not\leq_{im}$



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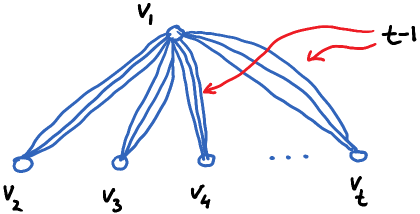

(instead of parallel edges we may have arbitrary edge-disjoint paths)

Rough Structure Theorem

Laminar family of edge-cuts: no two of the cuts cross.

Theorem [DMMS / Wollan]
If $K_t \not\leq_{im} G \Rightarrow \exists$ laminar family of edge-cuts, each of size $< (t-1)^2$ s.t. every block of the resulting vertex partition has at most $t-1$ vertices.

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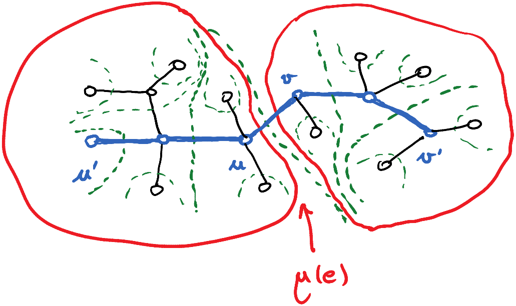
This is a rough characterization because a graph with the stated separation property cannot contain K_{t^2}-immersion.

Gomory-Hu Tree

Theorem: $\forall G : \exists$ tree H with $V(H) = V(G)$ and $\exists\, \mu : E(H) \to \mathbb{Z}$ s.t.

ⓐ $\forall e = uv \in E(H)$: $\mu(e) = \lambda(u, v)$

ⓑ $\forall u', v' \in V(G)$: $\lambda(u', v') = \min\limits_{e \in P_{u'v'} \subseteq H} \mu(e)$.



Proof of the Structure Theorem

- Gomory–Hu tree
- Remove edges with $\mu(e) < (t-1)^2$
- If a block with $\geq t$ vertices $v_1, v_2, \ldots, v_t$ remains:
    - $(t-1)^2$ edge-disjoint paths from v_1 to $\{v_2, \ldots, v_t\}$ with precisely $t-1$ of them ending at each v_i $(i \geq 2)$.
    - K_t-immersion by Example 2.

Some further results

- Abu-Khzam and Langston conjecture is true for $\chi(G) \leq 7$ [DeVos et al. 2010]
- (Lescure and Meyniel / DeVos, Kawarabayashi, M., Okamura) Every (simple) graph of minimum degree at least $k-1$ contains a K_k-immersion for $k \leq 7$
- For $k \geq 10$ no longer true (Seymour): K_{12} minus $E(4K_3)$. Generalized examples:
  $H_1, \ldots, H_r$ D-regular graphs, each with chromatic index $D+1$, where $r > \frac{1}{2}D(D+1)$. G complement of $H_1 \cup \cdots \cup H_t$, $n = |V(G)|$. Then $\delta(G) = n - D - 1$, but G does not contain K_{n-D}-immersion.

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- (DeVos, Dvořák, Fox, McDonald, M., Scheide) Every simple graph of minimum degree at least $200k$ contains a K_k-immersion

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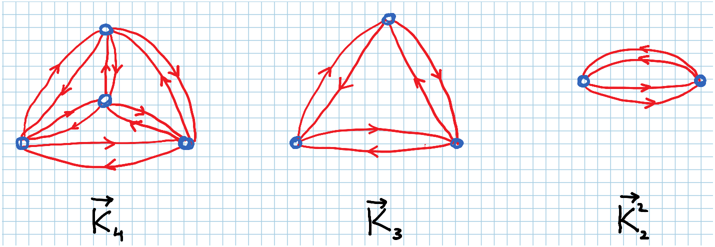
- And some good news:
  Tournaments are WQO (Chudnovsky and Seymour 2011)
- Eulerian digraphs of (out)degree ≤ 2 are WQO (Thesis of Thor)

Some more bad news

Complete digraph $\vec{K}_n$



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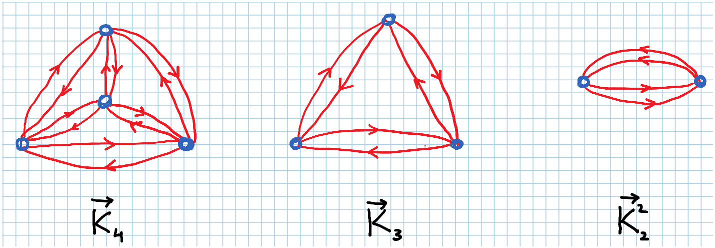
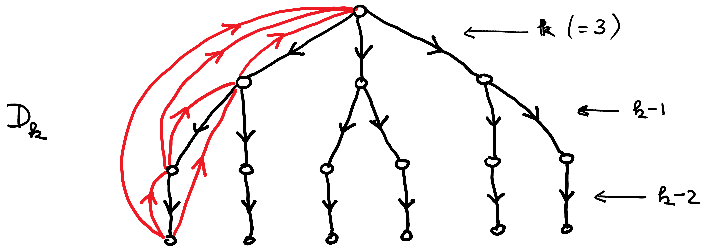


Theorem: For every positive integer k there exists a simple digraph D with minimum in- and outdegree at least k so that D does not immerse $\vec{K}_2^2$ (and hence does not immerse $\vec{K}_3$).



(1) $\forall v: \ \deg^+(v) = k$

(2) $\forall u \neq v: \ \lambda(u,v) = 1$ or $\lambda(v,u) = 1$

(3) $\forall v: \ \exists$ edge-disjoint paths $P_1, \ldots, P_k$ from v to the out-neighbors of the root.

Strong connectivity helps

Strongly k-edge-connected:
$D - S$ is strongly connected $\forall S \subseteq E(D)$ with $|S| < k$.
Arborescence with root v:
Spanning tree T of D s.t. all edges directed "away" from v

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Edmonds' Disjoint Arborescence Theorem: $v_1, \ldots, v_\ell \in V(D)$ (not necessarily distinct). Then there exist edge-disjoint arborescences $T_1, \ldots, T_\ell$ so that T_i has root v_i if and only if $\forall X \subset V(D)$:

$$d^+(X) \geq |\{i : v_i \in X, 1 \leq i \leq \ell\}|.$$

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Corollary: D strongly $t(t-1)$-edge-connected digraph with $|V(D)| \geq t \;\Rightarrow\; D$ contains an immersion of $\vec{K}_t$.

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Theorem: $\forall t \geq 3$ there exists a simple digraph which is strongly $\frac{1}{2}t(t-3)$-edge-connected and does not immerse $\vec{K}_t$.

Rough Structure – Eulerian Case

Theorem [DMMS] Let D be an Eulerian digraph. If $\vec{K}_t \not\leq_{im} D \Rightarrow$ $\exists$ laminar family of edge-cuts, each of size $< 2t(t-1)$ s.t. every block of the resulting vertex partition has at most $t-1$ vertices.

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This is a rough characterization because a digraph with the stated separation property cannot contain $\vec{K}_{t^2}$-immersion.

Proof is based on *ancient techniques*:

- Gomory-Hu tree on the undirected graph
- Remove edges with $\mu(e) < 2t(t-1)$
- If a part has a block with $\geq t$ vertices, split off the remaining vertices (Mader) and apply arborescence theorem

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The quadratic bound can be strengthened for small values of t as follows.

Theorem: For $t \leq 4$, every simple Eulerian digraph of minimum degree at least $t-1$ contains an immersion of $\vec{K}_t$.

Open for $t = 5$.